AN ELEMENTARY ESTIMATE FOR THE k-FREE INTEGERS[1]

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Communicated by George Whaples, December 24, 1962

1. **Introduction.** In this note k will denote a fixed integer >1. Let Q_k denote the sequence of k-free integers, that is, the integers whose prime factors are all of multiplicity $<k$. Also, let $\zeta(k)$ denote the sum of the series, $\sum_{n=1}^{\infty} n^{-k}$.

In this paper we prove the identity,

$$(1)\qquad \sum_{r=1}^{\infty}\left(\frac{\mu(r)}{J_k(r)}\right)c_k(n,r) = \begin{cases}\zeta(k) & \text{if } n\in Q_k,\\ 0 & \text{if } n\notin Q_k,\end{cases}$$

where $\mu(r)$ is the inversion function of number theory, $J_k(r)$ the Jordan totient of order k, and $c_k(n, r)$ is the generalized Ramanujan sum defined by (3) below. This is a special case of a much more general result proved in [**4**, Theorem 6]. In view of the intrinsic interest of the relation (1), an independent proof of its validity seems justified.

As a consequence of (1), we prove, without resorting to remainder estimates of series, the following asymptotic formula for the number $Q_k(x)$ of integers of Q_k not exceeding x:

$$(2)\qquad Q_k(x) = x/\zeta(k) + O(x^{1/k+\epsilon}),$$

for all $\epsilon>0$ (see Remark 2, §3). If one assumes an estimate for the remainder of the series $\sum_{n=1}^{\infty} n^{-s}$, $s>1$, (2) can be shown easily to hold with $\epsilon=0$ (see for example, [**7**, §18.6; **3**, §2]).

The method employed in the proof of (2) is essentially a generalization and refinement of a method introduced by Carmichael [**1**]. Carmichael obtained approximations for the average order of certain arithmetical functions using Ramanujan's trigonometric series expansions, in connection with an estimate involving Ramanujan's sum, $c(n, r)=c_1(n, r)$. The present discussion employs the more general sum $c_k(n, r)$ introduced by the author [**2**] and an appraisal for $c_k(n, r)$ which is sharper than the corresponding estimate of Carmichael (see (9) below).

2. **Proof of (1).** The function $c_k(n, r)$ is defined by

$$(3)\qquad c_k(n,r) = \sum_{a \,(\mathrm{mod}\ r);\ (a,r^k)_k=1} \exp(2\pi i a n/r^k),$$

[1] This research was supported in part by the National Science Foundation grant G-19012.