BOUNDED APPROXIMATION BY POLYNOMIALS[1]

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We announce a complete solution to the following problem. If G is an arbitrary bounded open set in the complex plane, which complex-valued functions in G can be obtained as the bounded pointwise limits in G of a sequence of polynomials?

THEOREM. *Given an arbitrary bounded open set G in the complex plane, and a complex-valued function f defined on G. There exists a sequence $\{p_n\}$ of polynomials that are uniformly bounded on G and that converge pointwise on G to f if and only if f has an extension F that is bounded and holomorphic on G^*, where G^* is the inside of the outer boundary of G.*

More precisely, G^* is the complement of the closure of the unbounded component of the complement of the closure of G.

In a certain sense, this result lies somewhere between Runge's theorem and Mergelyan's theorem [3]. The correct formulation of our theorem is in terms of sequences, and not topological closure. Indeed, for a certain bounded open set G, there exists a function f and functions f_n such that (i) each f_n is the bounded limit of a sequence of polynomials, (ii) f is the bounded limit of f_n, but (iii) f is not the bounded limit of any sequence of polynomials.

With G and G^* as above, we write $B_H(G)$ for the set of bounded holomorphic functions on G, and $B_H(G^*: G)$ for the set of functions on G that have a bounded holomorphic extension to G^*, and $P(G)$ for the set of functions on G that can be boundedly approximated on G by a sequence of polynomials. The theorem now reads:

$$P(G) = B_H(G^*: G).$$

Even if G is connected and simply connected, it may happen that G^* has several components. We define $G^\#$ as the union of those components of G^* that intersect G. Clearly, $B_H(G^*: G) = B_H(G^\#: G)$.

As a corollary to the theorem, we get a characterization of those bounded open sets G on which each bounded holomorphic function can be boundedly approximated by polynomials, namely $P(G) = B_H(G)$ if and only if $B_H(G) = B_H(G^*: G)$; in other words, if and only if the inner boundary of G is a set of removable singularities for all bounded holomorphic functions on G. The inner boundary of

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