LIE GROUP REPRESENTATIONS ON POLYNOMIAL RINGS[1]

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0. **Introduction.** 1. Let G be a group of linear transformations on a finite dimensional real or complex vector space X. Assume X is completely reducible as a G-module. Let S be the ring of all complex-valued polynomials on X, regarded as a G-module in the obvious way, and let $J\subseteq S$ be the subring of all G-invariant polynomials on X.

Now let J^+ be the set of all $f\in J$ having zero constant term and let $H\subseteq S$ be any graded subspace such that $S=J^+S+H$ is a G-module direct sum. It is then easy to see that

$$S = JH. \tag{0.1.1}$$

(Under mild assumptions H may be taken to be the set of all G-harmonic polynomials on X. That is, the set of all $f\in S$ such that $\partial f=0$ for every homogeneous differential operator ∂ with constant coefficients, of positive degree, that commutes with G.)

One of our main concerns here is the structure of S as a G-module. Regard S as a J-module with respect to multiplication. Matters would be considerably simplified if S were free as a J-module. One shows easily that S is J-free if and only if $S=J\otimes H$. This, however, is not always the case. For example S is not J-free if G is the two element group $\{I, -I\}$ and $\dim X\geqq 2$. On the other hand one has

EXAMPLE 1. It is due to Chevalley (see [2]) that if G is a finite group generated by reflections then indeed $S=J\otimes H$. Furthermore the action of G on H is equivalent to the regular representation of G.

EXAMPLE 2. S is J-free in case G is the full rotation group (with respect to some Euclidean metric on X. For convenience assume in this example that $\dim X\geqq 3$). Note that the decomposition of a polynomial according to the relation $S=J\otimes H$ is just the so-called "separation of variables" theorem for polynomials. This is so because J is the ring of radial polynomials and H is the space of all harmonic polynomials (in the usual sense).

Now, for any $x\in X$, let $O_x\subseteq X$ denote the G-orbit of x and let $S(O_x)$ be the ring of all functions on O_x defined by restricting S to O_x. Since J reduces to constants on any orbit it follows that (0.1.1) in-

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