POWER SERIES WITH INTEGRAL COEFFICIENTS

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Let $f(z)$ be a function, meromorphic in $|z|<1$, whose power series around the origin has integral coefficients. In [5], Salem shows that if there exists a nonzero polynomial $p(z)$ such that $p(z)f(z)$ is in H^2, or else if there exists a complex number α, such that $1/(f(z)-\alpha)$ is bounded, when $|z|$ is close to 1, then $f(z)$ is rational. In [2], Chamfy extends Salem's results by showing that if there exists a complex number α and a nonzero polynomial $p(z)$, such that $p(z)/(f(z)-\alpha)$ is in H^2, then $f(z)$ is rational. In this paper we show that if $f(z)$ is of bounded characteristic in $|z|<1$ (i.e. the ratio of two functions, each regular and bounded in $|z|<1$), then $f(z)$ is rational. If $f(z)$ is regular in $|z|<1$, then, by [4], $f(z)$ is of bounded characteristic in $|z|<1$, if and only if

$$\limsup_{r\to 1-} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta < \infty.$$

Thus any function in any H^p space ($p>0$) is of bounded characteristic. Hence, since the functions of bounded characteristic form a field, our result includes those of Salem and Chamfy.

Our first lemma gives a necessary condition for a function to be of bounded characteristic in $|z|<1$, in terms of the properties of its Taylor series coefficients.

If $g(z)=\sum_{i=0}^{\infty} a_i z^i$, we denote by $A_r=A_r(g)$ the matrix $\|a_{i+j}\|$, $0\leqq i, j\leqq r$.

LEMMA 1. *Suppose $g(z)$ is of bounded characteristic in $|z|<1$. Then $\det(A_r)\to 0$ as $r\to\infty$. More precisely,* $\lim_{r\to\infty} |\det(A_r)|^{1/r}=0$.

PROOF. By assumption, we may write $g(z)=s(z)/t(z)$, where $s(z)$ and $t(z)$ are bounded analytic functions in $|z|<1$. Suppose that $s(z)=\sum_{i=0}^{\infty} s_i z^i$ and $t(z)=\sum_{i=0}^{\infty} t_i z^i$, and, without loss of generality, that $t_0=1$. We now perform a series of column and row operations on the matrix A_r. Denote its columns from left to right by c_0, c_1, $c_2, \cdots, c_r$. Now, successively, for $j=0, 1, 2, \cdots, r$, replace the column c_{r-j} by $\sum_{i=0}^{r-j} t_i c_{r-j-i}$; then perform the same sequence of operations on the rows. This yields a matrix $D_r=\|d_{mn}\|$, $0\leqq m, n\leqq r$. Since $t_0=1$, $\det(D_r)=\det(A_r)$.