# QUASICONFORMAL MAPPINGS IN SPACE

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**1. Introduction.** Suppose that $w(z)$ is a homeomorphism of a plane domain $D$ onto a plane domain $D'$. We introduce, for convenience, the following functions which measure how much infinitesimal circles and areas are distorted under the mapping $w(z)$ at each point $z_0$ in $D$:

$$(1) \qquad H(z_0) = \limsup_{r\to 0} \frac{L(z_0, r)}{l(z_0, r)}, \qquad J(z_0) = \limsup_{r\to 0} \frac{m(U')}{m(U)}.$$

Here

$$L(z_0, r) = \sup_{|z-z_0|=r} |w(z) - w(z_0)|, \quad l(z_0, r) = \inf_{|z-z_0|=r} |w(z) - w(z_0)|,$$

$U'$ denotes the image of $U$, the disk $|z-z_0|<r$, and $m$ denotes Lebesgue plane measure. If $w(z)$ is differentiable at $z_0$, then $w(z)$ is locally affine at $z_0$ and maps the infinitesimal circles $|z-z_0|=\epsilon$ onto infinitesimal ellipses; $H(z_0)$ gives the ratio of the major to minor axes and $J(z_0)$ is the absolute value of the Jacobian.

Suppose next that $w(z)$ is continuously differentiable with $J(z)>0$ everywhere in $D$. Then $w(z)$ is said to be a *$K$-quasiconformal mapping*, in the classical sense, if $w(z)$ satisfies the dilatation condition

$$(2) \qquad H(z) \leq K, \qquad 1 \leq K < \infty,$$

everywhere in $D$. A homeomorphism is said to be quasiconformal if it is $K$-quasiconformal for some $K$. If $w(z)$ is sense-preserving with $H(z)=1$ in $D$, then the real and imaginary parts of $w(z)$ satisfy the Cauchy-Riemann equations and $w(z)$ is an analytic function of $z$. Hence a sense-preserving 1-quasiconformal mapping is conformal in the ordinary sense.

Quasiconformal mappings arise very naturally in complex function theory, for example in the study of multiply connected domains and in the Teichmüller problem. They are encountered also in the theory of partial differential equations as the univalent solutions of Beltrami systems. Finally, the study of such mappings is interesting in its own right, for though the theory usually parallels that of conformal map-

An address delivered before the Milwaukee meeting of the Society on November 17, 1961, by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings; received by the editors November 15, 1962.

[1] This address was prepared under a grant from the U. S. National Science Foundation, Contract NSF-G-18913.