AN "ALTERNIERENDE VERFAHREN" FOR GENERAL POSITIVE OPERATORS[1]

BY GIAN-CARLO ROTA[2]

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1. **Introduction.** Two types of limiting processes involving linear operators are of frequent occurrence. One type is associated with ergodic limits of Cesaro averages $n^{-1}(1+T+\cdots+T^{n-1})f$ of a linear operator T, for f in suitable function spaces, the so-called "ergodic theorems." The strongest such results establish almost everywhere convergence for f in a suitable Lebesgue space, under suitable hypotheses on T. This type of theorem is, nowadays, fairly well understood.[3]

A second type of limiting theorem of frequent occurrence concerns the limit of *products* $T_1T_2\cdots T_nf$ of a sequence of operators T_n. The two noteworthy special cases are (a) Limiting theorems of the type $\lim_{t\to\infty} P^tf$, where P^t is a semigroup of selfadjoint operators. Results of this type express the "tendency to equilibrium" in certain physical processes (typically, in diffusion theory). (b) One considers two or more noncommuting projections (most commonly *conditional expectations* in a probability space) say F_1 and F_2, and asks for $\lim_{n\to\infty}(F_1F_2)^nf$. This is an abstract rendering of the so-called *"alternierende Verfahren,"* (see [8]) which originated in function theory and has recently merged with various probabilistic and other considerations.

While results relating to the *mean convergence* of such iterations are easily obtained (for (a) and (b) they were obtained independently by von Neumann and Wiener [10; 15] in the 1930's), the corresponding statements relating to *almost everywhere* convergence have been missing. A result in this direction has recently been obtained by Burkholder and Chow [1], but their results, though more general in some respects, are limited to L_2.[4]

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[3] For the most refined results see [2; 7].

[4] Another important set of results has been recently obtained by E. M. Stein (personal communication, whose methods are of an altogether different character from the present ones; to him are due the statement and a first proof Theorem 2b below.