AN ELEMENTARY THEOREM IN GEOMETRIC INVARIANT THEORY

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The purpose of this note is to prove the key theorem in a construction of the arithmetic scheme of moduli M of curves of any genus. This construction, which relies heavily on Grothendieck's whole theory of schemes, may be briefly outlined as follows: first one defines the family K of tri-canonical models of curves C of genus g, any characteristic, in $\mathbf{P}^{5g-6}$, as a sub-scheme of one of Grothendieck's Hilbert schemes [3]. Second, one maps $K \to A$, where A (a sub-scheme of another Hilbert scheme) parametrizes the full projective family of the polarized Jacobians $J \subset \mathbf{P}^N$ of these curves. It may then be shown that M should be the orbit space $K/\mathrm{PGL}(5g-6)$; and it can also be shown that this orbit space exists if the orbit space $A/\mathrm{PGL}(N)$ exists. But, in general, given any family A of polarized abelian varieties $V \subset \mathbf{P}^N$ invariant under projective transformations, $A/\mathrm{PGL}(N)$ does exist; for simplicity assume that a section serving as an identity is rationally defined in the whole family A. Then A may be identified with the family of 0-cycles $\mathfrak{A} \subset \mathbf{P}^N$ which are the points of order m (suitable m) on the abelian varieties V. Now neglecting for simplicity the group of permutations of the m^{2g} points of these 0-cycles, this reduces the problem to constructing the orbit space $(\mathbf{P}^N)^{m^{2g}}/\mathrm{PGL}(N)$. In this paper, an apparently very natural open sub-scheme[1] $(\mathbf{P}^n)_0^m \subset (\mathbf{P}^n)^m$, any n, m, is constructed such that in fact $(\mathbf{P}^n)_0^m$ is a principal fibre bundle over its quotient by $\mathrm{PGL}(n)$. This result is apparently new even over the complex numbers. The methods used are entirely elementary, and no special techniques are used to deal with the generalization from varieties to schemes. Moreover, except for the replacement of $\mathbf{Z}$ by k, no changes are necessary or appear possible should the reader wish to consider the objects as varieties rather than schemes.

1. Let $\mathbf{P}^n$ be projective n-space over $\mathbf{Z}$, $(\mathbf{P}^n)^m$ the m-fold product with itself. Let homogeneous coordinates in the ith factor be $X_0^{(i)}$, $X_1^{(i)}, \cdots, X_n^{(i)}$, and for all $(n+1)$-tuples $i_0, i_1, \cdots, i_n$ let

$$D_{i_0,i_1,\cdots,i_n} = \operatorname*{Det}_{0 \leq k,l \leq n} (X_l^{(i_k)}).$$

[1] See also [2] for a slightly larger open sub-scheme.