# A REPRESENTATION OF THE INFINITESIMAL GENERATOR OF A DIFFUSION PROCESS[1]

BY L. L. HELMS

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0. **Introduction.** Let $\Omega$ be a connected locally compact metric space and let $C(\Omega)$ denote, as usual, the Banach space of bounded continuous real functions on $\Omega$. A diffusion process (see [1] for definitions) is a semi-group $\{T_t; t>0\}$ of positivity preserving bounded linear transformations on $C(\Omega)$ which is strongly continuous for $t>0$. Such semi-groups are also required to be of local character; i.e., if $x$ vanishes in a neighborhood of a point $\xi \in \Omega$, then

$$Ax(\xi) = \lim_{t \downarrow 0} \frac{T_t x - x}{t}(\xi) = 0.$$

Consider an arbitrary element $x \in C(\Omega)$. If (i) the above limit exists for all $\eta$ in a neighborhood $W$ of $\xi$, (ii) the convergence is bounded on this neighborhood, and (iii) $Ax$ is continuous on $W$, then $x$ is said to be in the local domain of the operator $A$ at $\xi$. $x$ is said to be in the global domain, $D(A)$, of the operator $A$ whenever $W = \Omega$. Feller (see [1]) has posed the problem of characterizing the operator $A$. A local representation of such operators will be discussed in this note.

One of the essential properties of the operator $A$ is the maximum property; i.e., $Ax(\xi) \leqq 0$ whenever $x$ is in the local domain of $A$ at $\xi$ and $x$ has a null maximum at $\xi$. Before discussing the representation of $A$, a few remarks concerning the denseness in $C(\Omega)$ of the global domain of the operator $A$ are in order. A null point of $A$ is a point $\xi \in \Omega$ such that $Ax(\xi) = x(\xi) = 0$ for all $x$ in the local domain of $A$ at $\xi$. Feller has shown that the set $N$ of null points is a closed set. He has also shown that if $x$ vanishes outside a compact set which does not meet $N$, then there is a sequence $X_\lambda$ in the global domain of $A$ such that $X_\lambda \rightarrow x$ strongly [1]. Using this result, one can show that $D(A)$ is locally dense in $C(\Omega)$ at each point $\xi \in \Omega - N$; i.e., if (i) $\xi \in \Omega - N$, (ii) $W$ is a neighborhood of $\xi$ such that $\overline{W} \subset \Omega - N$, and (iii) $x \in C(\Omega)$, then $x$ can be approximated uniformly over $\overline{W}$ by an element of $D(A)$.

Another type of point at which the operator $A$ may be degenerate is the absorption point; i.e., $\xi$ is an absorption point if there is a real

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