ON THE EXISTENCE OF INVARIANT MEASURES

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Let T be a measurable transformation on a measure space $(\Omega, \mathcal{A}, P)$, with $0 < P(\Omega) < \infty$. Call T absolutely continuous if $P(A) = 0$ implies $P(T^{-1}A) = 0$. The transformation T is said to have the Birkhoff recurrence property if, for each $A \in \mathcal{A}$, $\lim_{n\to\infty}(1/n)\sum_{j=0}^{n-1}\chi_A(T^j\omega)$ exists for almost all $\omega \in \Omega$. It has been shown that if T is absolutely continuous and has the Birkhoff recurrence property, then there exists a non-negative, finite, countably additive measure Q on $\mathcal{A}$ such that (i) $Q \ll P$, (ii) Q and P agree on invariant sets, (iii) $Q(A) = Q(T^{-1}A)$ for each $A \in \mathcal{A}$ [3]. In this paper we prove the following result.

THEOREM. *If T is an absolutely continuous measurable transformation on $(\Omega, \mathcal{A}, P)$, where $0 < P(\Omega) < \infty$, then there exists a non-negative, finite, finitely additive measure Q with the following properties*: (i) $P(A) = 0$ *implies* $Q(A) = 0$; (ii) *Q and P agree on invariant sets*; (iii) $Q(A) = Q(T^{-1}A)$ *for each* $A \in \mathcal{A}$.

We shall only outline the proof here. Let $\mathcal{B}$ be the collection of all invariant sets; that is, $B \in \mathcal{B}$ if and only if $B = T^{-1}B$. Then $\mathcal{B}$ is a σ-subalgebra of $\mathcal{A}$. Consider the real algebras $L^\infty(\mathcal{A})$ and $L^\infty(\mathcal{B})$, and represent them as the algebras $R(X)$ and $R(Y)$, respectively, of all continuous real-valued functions on the extremally disconnected, compact, Hausdorff spaces X and Y. The Boolean algebras $E(X)$ and $E(Y)$ of idempotents in $R(X)$ and $R(Y)$ are both complete. Moreover there is a natural isomorphism of $R(Y)$ into $R(X)$ which maps $E(Y)$ into $E(X)$. The dual of this is a continuous mapping π of X onto Y, and the completeness of $E(Y)$ assures that the mapping is an open mapping. Theorems of Gleason [1] and Halmos [2] assert that π has many cross-sections.

For any $f \in R(X)$ define functions Mf and mf on Y by setting $Mf(y) = \text{lub}\{f(x) : \pi x = y\}$ and $mf(y) = \text{glb}\{f(x) : \pi x = y\}$. Since π is open, both Mf and mf are in $R(Y)$. Call a linear transformation $\mu: R(X) \to R(Y)$ a *generalized mean* if $mf \leq \mu f \leq Mf$ for each $f \in R(X)$. Every cross-section of π gives a mean that is a homomorphism. An absolutely continuous T induces a linear transformation t of $R(X)$ into itself. A generalized mean μ will be called invariant if $\mu tf = \mu f$ for each $f \in R(X)$. The set of all means is a nonempty, compact,

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