SPACES OF MEASURABLE TRANSFORMATIONS[1]

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By a *space* we shall mean a measurable space, i.e. an abstract set together with a σ-ring of subsets, called *measurable* sets, whose union is the whole space. The *structure* of a space will be the σ-ring of its measurable subsets. A *measurable transformation* from one space to another is a mapping such that the inverse image of every measurable set is measurable.

Let X and Y be spaces, F a set of measurable transformations from X into Y, and ϕ_F: $F \times X \to Y$ the natural mapping defined by $\phi_F(f, x) = f(x)$. A structure R on F will be called *admissible* if ϕ_F, considered as a mapping from the product space $(F, R) \times X$ into Y, is a measurable transformation.[2] It may not be possible to define an admissible structure on F; if it is, F itself will also be called *admissible*. We are concerned with the problem of characterizing, for given X and Y, the admissible sets F and the admissible structures R on the admissible sets.

The following three theorems may be established fairly easily:

THEOREM A. *A set consisting of a single measurable transformation is admissible.*

THEOREM B. *A subset of an admissible set is admissible. Indeed, if $G \subset F$, R is an admissible structure on F, and R_G is the subspace structure on G induced[3] by R, then R_G is admissible on G.*

THEOREM C. *The union of denumerably many admissible sets is admissible. Indeed, if $F = \bigcup_{i=1}^{\infty} F_i$ and $R_1, R_2, \cdots$ are admissible structures on $F_1, F_2, \cdots$ respectively, then the structure R on F generated by the members of all the R_i is admissible on G.*

Much more can be said if X and Y are assumed to be *separable*, i.e. to have countably generated structures.[4] To state our theorems in this case we first define the concept of Banach class, closely related to that of Baire class. Let $\mathfrak{A}$ be an arbitrary class of meas-

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[1] The author is much indebted to Professor P. R. Halmos, who suggested a number of significant improvements in the complete version of this note.

[2] (F, R) is the space whose underlying abstract set is F and whose structure is R.

[3] R_G consists of all intersections of G with members of R.

[4] The term is used by analogy with its topological use. We will also use the term "separable structure," meaning a countably generated structure.