ON INVARIANT MEASURES

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Given a measurable transformation on a measure space one can ask whether or not there is an equivalent measure that is invariant under the transformation. This problem is discussed very thoroughly in Halmos' *Lectures on ergodic theory*, pp. 81–90, 97. The first result along these lines is due to E. Hopf who obtained necessary and sufficient conditions for the existence of a finite invariant measure. The condition is that the whole space is "bounded," i.e. that the space is not a "copy" of a subset of strictly smaller measure. ("Copy" is defined below.) Recently Hajian and Kakutani (the paper is not yet published) showed that Hopf's condition is equivalent to the nonexistence of a set of nonzero measure having infinitely many disjoint images under the powers of the transformation. In [**3**] Halmos proved that there was a sigma-finite invariant measure if and only if the space was the union of a countable number of "bounded" sets. It was not known however whether or not every transformation had this property. Our example shows that there are transformations that admit no equivalent invariant measures.

THEOREM. *There exists a* 1-1 *invertible measurable and nonsingular transformation, T, on the unit interval such that there is no sigma-finite measure equivalent to Lebesgue measure which is invariant under T.*

We could modify the example a little so that the only invariant measure which is absolutely continuous with respect to Lebesgue measure is identically 0.

(A 1-1 invertible, measurable, nonsingular transformation is one such that it and its inverse take measurable sets into measurable sets and sets of measure 0 into sets of measure 0. A measure equivalent to Lebesgue measure is a measure which is defined on the same class of measurable sets and has the same sets of measure 0. Sigma-finite means that the interval is the union of a countable number of sets of finite measure.

It is possible to have a sigma-finite measure equivalent to Lebesgue measure such that every interval has infinite measure and it is this sort of thing that complicates our construction.)

A transformation T on the unit interval will be said to have property P if: for any integer N and any set S of Lebesgue measure $>9/10$ there is a set $M \subset S$ of Lebesgue measure $1/8$ such that there