COMMUTATIVE SUBGROUPS AND TORSION IN COMPACT LIE GROUPS

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In this note, G is a compact connected Lie group. We are concerned with the torsion of the cohomology ring $H^*(G; Z)$ of G over the integers, certain commutative subgroups of G, and relations between these two questions.

NOTATION. $E(m_1, \cdots, m_r)$ or $E_A(m_1, \cdots, m_r)$ denotes the exterior algebra over the ring A of a free A-module with r generators of respective degrees $m_1, \cdots, m_r$; p is a prime number, Z_p the field of integers mod p, Q the field of rational numbers. Tors H is the torsion subgroup of a group H, ord M the order of a finite group M. The identity component of a closed subgroup H if G is denoted by H_0, if L is a subset of G, its centralizer in G is denoted by $Z(L)$.

1. **H-spaces with finitely generated cohomology groups.** In this section, X is a compact connected H-space, for which $H^*(X; Z)$ is finitely generated. As is known $H^*(X; Q) = E(m_1, \cdots, m_r)$ with m_i odd; we assume $m_i \leqq m_j$ if $i \leqq j$. With this notation we have

PROPOSITION 1.1. (a) $H^*(X; Z)/\text{Tors } H^*(X; Z) = E_Z(m_1, \cdots, m_r)$. (b) *Let p be odd and K be a field of characteristic p. Then $H^*(X; K)$ contains a subalgebra isomorphic to $E_K(m_1, \cdots, m_r)$. Each system of generators of type (M) of $H^*(X; K)$ (in the sense of* [**2**, §6]) *contains at least r elements of odd degrees.*

Let $H = H^*(X; Z)/\text{Tors } H^*(X; Z)$. The transpose of the product map induces a homomorphism $H \to H \otimes H$ satisfying the conditions imposed on a Hopf algebra over Z. Hence $H \otimes A$ is a Hopf algebra over A for any ring A. Then (a) follows from the structure theorem [**2**, Théorème 6.1] applied to $H \otimes L$, L being a field, by an easy induction. (b) follows from (a), the structure theorem, and the fact that the product of the generators of $E_K(m_1, \cdots, m_r)$ is nonzero.

PROPOSITION 1.2. *Let p be odd. Assume that each element of $H^*(X; K_p)$ has height $\leqq p$ (in the sense of* [**2**, §6]) *and that X is simply connected. Let k be the first integer such that $H^k(X; Z)$ has p-torsion. Then $p \cdot k \leqq m_r + p - 1$.*

This may be proved using the spectral sequence connecting $H^*(X; Z_p)$ to $H^*(X; Z)/\text{Tors } H^*(X; Z) \otimes Z_p$, whose differentials are the successive Bockstein operators. This result is sufficient for the