APPROXIMATION OF SMOOTH FUNCTIONS

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Theorems of Jackson and S. Bernstein about the approximation of smooth functions are usually interpreted in the way that all functions with a prescribed degree of smoothness have a definite degree of approximation. They can be viewed in another way, which reveals their susceptibility to generalization.

Let $\omega(h)$ be an increasing continuous subadditive function defined for $h \geqq 0$ with $\omega(0)=0$, A a compact metric space with infinitely many points. By C_1^ω we denote the set of all real valued functions f on A with $|f(x)| \leqq 1$, $|f(x)-f(x')| \leqq \omega(h)$, $h=\rho(x, x')$. If A is a q-dimensional cube, p a natural number and $0<\alpha \leqq 1$, we denote by $C_1^{p+\alpha}$ the set of all functions on A with continuous partial derivatives of orders not exceeding p and bounded by 1, and with the derivatives of order p satisfying a Lipschitz condition of order α and with coefficient 1. Let $G=\{g_n\}$ be a sequence of continuous functions on A. Then, with some norm, for example the uniform norm on A,

$$E_n(f) = E_n^G(f) = \inf \left\| f - \sum_{i=1}^n a_i g_i \right\|$$

is the degree of approximation of f by linear combinations of $g_1, \cdots, g_n$; and

$$\mathcal{E}_n(W) = \sup_{f \in W} E_n(f)$$

is the degree of approximation of a class W.

The theorems of Jackson and Bernstein state that for periodic $f \in C_1^{p+\alpha}$, and the trigonometric approximation, $E_n(f)$ has the exact order $n^{-(p+\alpha)/q}$; exceptions occur only if f has a higher degree of smoothness. We regard this as a statement about a certain massivity of $C_1^{p+\alpha}$, which prevents better approximation by linear combinations of only n functions. One can hope that an estimate of $\mathcal{E}_n(C_1^{p+\alpha})$ from below can be given for an arbitrary system G, and that the trigonometric system is close to the best possible. That this is true, is shown by the following results:

THEOREM 1. *Let A be a compact metric space, and $\delta=\delta(n)$ the largest*

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