Stochastic processes. By J. L. Doob. New York, Wiley, 1953. 8+654 pp. $10.00.

Chapter 1. Introduction and probability background (45 pages). The fundamental concepts of conditional probability and expectation are generalized over and somewhat different from those defined in Kolmogorov's *Grundbergriffe* (1933). Let y be a random variable whose expectation exists: let $\mathcal{J}$ be a Borel field of measurable ω sets and $\mathcal{J}'$ its "completion." The conditional expectation of y relative to $\mathcal{J}$, denoted by $E\{y|\mathcal{J}\}$, is any measurable $\mathcal{J}'$, integrable ω function which satisfies $\int_\Lambda E\{y|\mathcal{J}\}dP=\int_\Lambda y dP$ for every $\Lambda\in\mathcal{J}$. If $\mathcal{J}$ is the smallest Borel field $\mathcal{B}(x_t, t\in T)$ with respect to which the x_t's are measurable, the above definition specializes to the conditional expectation of y with respect to the x_t's. The conditional probability of a measurable set M, denoted by $P(M|\mathcal{J})$, is defined to be $E\{y|\mathcal{J}\}$ where $y(\omega)$ is the characteristic function of M. An important question arises: Is it possible to define an M, ω function $P(M, \omega)$ such that (i) for every ω, $P(\cdot, \omega)$ is a probability measure of M and for every M, $P(M, \cdot)$ is measurable $\mathcal{J}'$; (ii) for every M, $P(M, \cdot)=P\{M|\mathcal{J}\}$ with probability one. If such a $P(\cdot, \cdot)$ is defined for every $M \in\mathcal{B}(y_1, \cdots, y_n)$, it is called the conditional probability distribution of the y_j's relative to $\mathcal{J}$. A related question is as follows. Let Y denote a generic n-dimensional Borel set. Is it possible to define a Y, ω function $p(Y, \omega)$ such that (i) for every ω, $p(\cdot, \omega)$ is a probability measure of Y and for every Y, $p(Y, \cdot)$ is measurable $\mathcal{J}'$; (ii) for every Y, $p(Y, \omega)=P\{[y_1(\omega), \cdots, y_n(\omega)]\in Y\}$ with probability one. If such a $p(\cdot, \cdot)$ exists, it is called the conditional probability distribution of $y_1, \cdots, y_n$ in the wide sense relative to $\mathcal{J}$. Now the main result is: while a conditional probability distribution in the wide sense always exists, a conditional probability distribution may fail to exist. The point is that an ω set $M\in\mathcal{B}(y_1, \cdots, y_n)$ does not uniquely determine the Borel set Y and if $M=\{[y_1(\omega), \cdots, y_n(\omega)]\in Y\}$ is satisfied for $Y=Y_1$ and also for $Y=Y_2$ it does not necessarily follow that $p(Y_1, \omega)=p(Y_2, \omega)$ with probability one. A sufficient condition for this, and so for the existence of a $P(M, \omega)$, is e.g. that the range of $[y_1(\omega), \cdots, y_n(\omega)]$ be a Borel set; this condition is always satisfied if the y process is of "function space type" (see below). The lack of a conditional probability distribution however vitiates a couple of the author's more famous theorems (1938) (see the Bibliography for all references). A correct form of the extension theorem, due to Ionescu Tulcea, where the existence of conditional probability distributions is assumed, is given in the Appendix. Kolmogorov's version is not given explicitly but is implied by this theorem and the sufficient condition