BOUNDED ANALYTIC FUNCTIONS

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If D is a domain in the complex z-plane, then the family $B = B(D)$ of bounded analytic functions in D is defined as consisting of those analytic functions $f(z)$ which are regular and single-valued in D and which satisfy the inequality $|f(z)| < 1$ at all points of D. The classical investigations of the family $B(D)$ were restricted to the case in which D is a simply-connected domain. In fact, D was generally taken to be the interior of the unit circle, a restriction which is apparent rather than real since most properties of bounded functions are either invariant with respect to a conformal mapping of D, or else are transformed in a simple manner. The use of the simple properties of the unit circle led to a large number of results which are distinguished both by their elegance and their preciseness. However, since the proofs leading to these results lean heavily on the special features of the unit circle, they gave little or no indication as to their possible generalization to the case of bounded functions in multiply-connected domains.

In the classical treatment of bounded functions, the family B was occasionally replaced by the more general class of analytic functions $w = f(z)$ whose values—for $z \in D$—are contained in a specified simply-connected domain D' in the w-plane. The family B corresponds to the case in which D' is the unit circle $|w| < 1$. Other special cases are the family of functions with a positive real part—to be denoted by $P = P(D)$—obtained if D' is the right half-plane $\operatorname{Re}\{w\} > 0$, and the family of functions with a bounded real part—denoted by $B_R = B_R(D)$—for which D' is the infinite strip $-1 < \operatorname{Re}\{w\} < 1$. These families are obtained from B by means of the conformal transformations which carry $|w| < 1$ into the various domains D'. For instance, we have

$$g(z) = \frac{1 + f(z)}{1 - f(z)}, \qquad f(z) \in B, \; g(z) \in P, \tag{1}$$

$$\phi(z) = \frac{4}{\pi} \arc \tan f(z), \qquad f(z) \in B, \; \phi(z) \in B_R. \tag{2}$$

Apart from their intrinsic interest, these classes are often useful in the investigation of the functions of B since the special features of

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