# A NEW PROOF OF E. CARTAN'S THEOREM ON THE TOPOLOGY OF SEMI-SIMPLE GROUPS

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1. **Introduction.** E. Cartan has proved[2] that a connected semi-simple Lie group is topologically the direct product of a compact subgroup and a Euclidean space. Cartan first proved this theorem in 1927 by a reduction to special cases, and not until 1929 did he free his proof from the consideration of special cases. As a result Cartan's proof is diffused among several journals. Moreover, Cartan employs in an essential way the theory of symmetric Riemannian spaces and makes use of a result whose proof seems to be lacking (Ann. École Norm. loc. cit. p. 367, §20).

In this paper there will be given a more direct proof which eliminates the use of symmetric Riemannian spaces. The author wishes to acknowledge his debt to Professor C. Chevalley who suggested in an oral communication Lemma 1.4 below and to whom a proof of Theorem 1, essentially the same as the one given here, was known.

2. **Definitions and preliminaries.** Let $\mathcal{G}$ be a semi-simple Lie algebra over a field $K$ of characteristic zero. Let ad $g$ denote the linear transformation $x \to [g, x]$, where $x, g \in \mathcal{G}$. By a Cartan subalgebra is meant a subalgebra $\mathcal{H}$ of $\mathcal{G}$ maximal with respect to properties (1) $\mathcal{H}$ is abelian, that is, $[\mathcal{H}, \mathcal{H}] = 0$; and (2) $H \in \mathcal{H}$ implies ad $H$ is semi-simple, that is, its minimal equation has no repeated factor. It is a theorem that a Cartan subalgebra is a maximal abelian subalgebra (but not conversely).

Let $\mathcal{G}$ be a semi-simple Lie algebra, that is, $\mathcal{G}$ contains no abelian ideal, and let $\mathcal{H}$ be any Cartan subalgebra. Assume that the base field $K$ is algebraically closed. A nonzero linear function $\alpha$ defined on $\mathcal{H}$ is called a "root" if and only if there exists an $X$ in $\mathcal{G}$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathcal{H}$. Any nonzero $Y$ in $\mathcal{G}$ with this property is said to "belong to $\alpha$."

The following is known:[3]

R1. If $X_\alpha$, $X_\beta$ belong to $\alpha$, $\beta$ respectively, then $[X_\alpha, X_\beta]$ belongs

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[2] Ann. École Norm. vol. 44 (1927) p. 367, cf. Bull. Soc. Math. France vol. 55 (1927) pp. 122–125, J. Math. Pures Appl. (9) vol. 8 (1929) pp. 24–27.

[3] Cf. H. Weyl, *Continuous groups*, vol. 2, Institute for Advanced Study Notes, 1935, pp. 69–87.