ON THE FREQUENCY OF PAIRS OF SQUARE-FREE NUMBERS WITH A GIVEN DIFFERENCE

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If k is a positive integer, then the function

$$f(x) = f(x, k) = \sum_{n \leq x} | \mu(n)\mu(n + k) |$$

enumerates the number of pairs of square-free integers with fixed difference k such that the smaller of the two does not exceed x. The purpose of the present note is to establish the following result.

THEOREM. *As* $x \rightarrow \infty$ *we have*

$$f(x) = \prod_{p} \left(1 - \frac{2}{p^2}\right) \prod_{p^2 | k} \left(1 + \frac{1}{p^2 - 2}\right) x + O(x^{2/3} \log^{4/3} x),$$

where the O-constant may depend upon k.

In a previous publication[1] I considered the more general sum

$$F(x) = \sum_{n \leq x} \mu_r(n + k_1) \cdots \mu_r(n + k_s),$$

where $k_1, \cdots, k_s$ are distinct integers, r is an integer greater than 1, and $\mu_r(n)$ is defined as 0 or 1 according as n is or is not divisible by the rth power of a prime. I showed that, for $x \rightarrow \infty$,

$$F(x) = Ax + O(x^{2/(r+1)+\epsilon}), \tag{1}$$

where A is a constant which can be expressed as an infinite series or else as a product ranging over primes. The asymptotic formula (1) generalized and sharpened an earlier estimate due to Pillai.[2] The present note furnishes a slight improvement on (1) for the case $r = 2$, $s = 2$. The factor x^ϵ in (1) arose from the expression $\max_{\nu \leq x} d(\nu)$, and could not, therefore, be replaced by a power of $\log x$ by the method previously used.

Our notation is as follows. The letters x, y denote positive numbers; all other small letters denote positive integers unless otherwise stated, and p is reserved for primes.

The O-notation refers to the passage $x \rightarrow \infty$, and O-constants de-

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[1] L. Mirsky, *Note on an asymptotic formula connected with r-free integers*, Quart. J. Math. Oxford Ser. vol. 18 (1947) pp. 178–182.

[2] S. S. Pillai, *On sets of square-free integers*, J. Indian Math. Soc. N.S. vol. 2 (1936) pp. 116–118.