CYCLIC INVARIANCE UNDER MULTI-VALUED MAPS[1]

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In what follows it is always assumed that X, Y are compact (=bicompact) connected Hausdorff spaces each containing more than one point.

Let f denote a function which assigns to each x in X a *subset* $f(x)$ of Y. We suppose that the sets $\{f(x)\}$ cover Y. By definition

$$f^{-1}(y) = \{x \mid x \in f(y)\}.$$

It is assumed that the sets $\{f^{-1}(y)\}$ cover X. The functions f and f^{-1} play dual roles inasmuch as $f=(f^{-1})^{-1}$. If f is single-valued, then f^{-1} is the inverse of f in the usual meaning of the term. For $A \subset X$, $B \subset Y$ we define

$$f(A) = \bigcup\{f(x) \mid x \in A\}, \qquad f^{-1}(B) = \bigcup\{f^{-1}(y) \mid y \in B\}.$$

When f is single-valued we know that continuity is equivalent to the assertion that A, B closed imply $f(A)$, $f^{-1}(B)$ closed. When f is multi-valued we take this as a *definition of continuity*. It does not follow, as in the single-valued case, that $f^{-1}(B)$ is open if B is open. These definitions include both a single-valued map (=continuous function) and its inverse.

In this note we show that certain theorems of analytic topology carry over to multi-valued maps (=continuous multi-valued functions as defined above). Some of our results are new even for single-valued maps. Except for fixed-point theorems there seem to be no results in the literature for multi-valued maps.

We say that f is *anarthric* if it is continuous and if for $y \in Y$ no $x \in X - f^{-1}(y)$ separates $f^{-1}(y)$ in X. If f is single-valued and non-alternating, then f is anarthric. See Wallace [2],[2] [3], and [4] and Whyburn [5] and [6]. It is clear that if f is the inverse of a single-valued map, then f is anarthric.

For simplicity we write $P|Q$ to mean that the sets P and Q are mutually separated. Also if p, $q \in X$, then $p \sim q$ means that no point separates p and q in X.

THEOREM 1. *In order that the multi-valued map f be anarthric each of the following conditions is both necessary and sufficient:*

Received by the editors June 28, 1948.

[1] This work was done under Contract N7-onr-434, Task Order 3, Navy Department, The Office of Naval Research.

[2] Numbers in brackets refer to the bibliography at the end of the paper.