A CONVEX METRIC FOR A LOCALLY CONNECTED CONTINUUM

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A topological space is metrizable if there is a distance function $D(x, y)$ such that if x, y, z are points, then

(1) $D(x, y) \geqq 0$, the equality holding only if $x = y$,

(2) $D(x, y) = D(y, x)$ (symmetry),

(3) $D(x, y) + D(y, z) \geqq D(x, z)$ (triangle condition),

(4) $D(x, y)$ preserves limit points.

By (4) we mean that x is a limit point of the set T if and only if for each positive number ϵ there is a point of T at a positive distance from x of less than ϵ. We say that the metric $D(x, y)$ is convex if for each pair of points x, y there is a point u such that

(5) $$D(x, u) = D(u, y) = D(x, y)/2.$$

A subset M of a topological space S is said to have a convex metric (even though S may have no metric) if the subspace M of S has a convex metric.

It is known [5][1] that a compact continuum is locally connected if it has a convex metric. The question has been raised [5] as to whether or not a compact locally connected continuum M can be assigned a convex metric. Menger showed [5] that M is convexifiable if it possesses a metric D such that for each point p of M and each positive number ϵ there is an open subset R of M containing p such that each point of R can be joined in M to p by a rectifiable arc of length (under D) less than ϵ. Kuratowski and Whyburn proved [4] that M has a convex metric if each of its cyclic elements does. Beer considered [1] the case where M is one-dimensional. Harrold found [3] M to be convexifiable if it has the additional property of being a plane continuum with only a finite number of complementary domains.

We shall show that if M_1 and M_2 are two intersecting compact continua with convex metrics D_1 and D_2 respectively, then there is a convex metric D_3 on M_1+M_2 that preserves D_1 on M_1 (Theorem 1). Using this result, we show that any compact n-dimensional locally connected continuum has a convex metric (Theorem 6). We do not

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[1] Numbers in brackets refer to the references cited at the end of the paper.