A BOUND FOR THE MEAN VALUE OF A FUNCTION

J. ERNEST WILKINS, JR.

Let $f(t)$ be a bounded measurable function defined when $0 \leqq t \leqq \pi$. The Fourier sine series associated with $f(t)$ is

$$\sum_{n=1}^{\infty} b_n \sin nt, \qquad b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt.$$

We shall be interested in this paper in establishing a bound for the mean value[1]

$$a = \frac{1}{\pi} \int_0^{\pi} f(t) dt$$

when $f(t)$ is such that one of the coefficients b_n vanishes.

We can suppose without essential loss of generality that $|f(t)| \leqq 1$. Since $b_{2n} = 0$ whenever $f(t)$ is constant, it is clear that the only conclusion on a that can be drawn from the inequality $|f(t)| \leqq 1$ and the equality $b_{2n} = 0$ is that $|a| \leqq 1$, and this conclusion is valid whether b_{2n} vanishes or not. Hence we shall restrict attention to b_{2n+1}. For the same reason we shall not discuss the vanishing of the coefficient

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt$$

of the Fourier cosine series of $f(t)$.

Suppose that $b_{2n+1} = 0$. Define a positive number y by the equation $y = \sin [\sec^{-1}(2n+2)]$ where the $\sec^{-1}$ lies between 0 and $\pi/2$. Let E be the sum of the intervals

$$\frac{2p\pi + \sin^{-1} y}{2n+1} \leqq t \leqq \frac{(2p+1)\pi - \sin^{-1} y}{2n+1} \qquad (p = 0, 1, \cdots, n),$$

where the $\sin^{-1}$ lies between 0 and $\pi/2$. Then it is clear that

$$(1) \qquad \begin{aligned} &\sin (2n+1)t \geqq y && \text{if } t \text{ is in } E, \\ &\sin (2n+1)t < y && \text{if } t \text{ is not in } E. \end{aligned}$$

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[1] The importance of the concept of mean value in the study of Fourier series can be seen by consulting Bohr [1, pp. 7–29]. Numbers in brackets refer to the bibliography at the end of the paper.