ON DESCARTES-HARRIOT'S RULE[1]

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The classical rule of Descartes-Harriot asserts that if

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \tag{1}$$

is a polynomial with real coefficients, then the number P of its positive zeros cannot exceed the number V_0 of the sign changes of the sequence

$$a_0, a_1, \cdots, a_n \tag{2}$$

(if some of the coefficients vanish they can be omitted). This rule has been extended and refined in various ways. V_0 yields obviously an upper bound for the number P^* of zeros lying in $0<x<1$; then Laguerre[2] observed the P^* is majorised also by the number V_1 of sign changes of the sequence

$$a_0, (a_0 + a_1), \cdots, (a_0 + a_1 + \cdots + a_n), (a_0 + \cdots + a_n), \cdots \tag{3}$$

which is not greater than V_0. More generally P^* is majorised by V_k, where V_k denotes the number of sign changes of that sequence which arises from (2), completing it with zeros to an infinite sequence and forming the sequence of k times iterated partial sums; these V_k's form a nonincreasing sequence. One trend of the investigations is the study of the V_k for large k; this was done mainly by M. Fekete and G. Pólya.[3] Another trend is to obtain similar rules representing $f(x)$ in various other forms (Runge,[4] Sylvester,[5] Obreschkoff,[6] I. J. Schoenberg[7]) or even generally for the linear combinations

$$h(x) = a_0\phi_0(x) + \cdots + a_n\phi_n(x)$$

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[1] To the memory of my late friend, Ervin Feldheim.

[2] E. Laguerre, *Mémoire sur la théorie des équations numériques*, J. Math. Pures Appl. (3) vol. 9 (1883) pp. 99–146.

[3] See in particular their joint paper in Rend. Circ. Mat. Palermo vol. 34 (1912) pp. 89–120 entitled *Über ein Problem von Laguerre.*

[4] See the paper of G. Pólya, *Über einige Verallgemeinerungen der Descartesschen Zeichenregel*, Archiv der Mathematik und Physik vol. 23 (1915) pp. 22–32.

[5] J. J. Sylvester, *Mathematical papers*, vol. 2, pp. 360 and 401.

[6] N. Obreschkoff, *Über die Wurzeln algebraischer Gleichungen*, J. Deutschen Math. Verein vol. 33 (1924) pp. 52–64.

[7] I. J. Schoenberg, *Zur Abzählung der reellen Wurzeln algebraischen Gleichungen*, Math. Zeit. vol. 38 (1934) pp. 546–564.