APPROXIMATION IN LIP (α, p)

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Let L_p, $1<p<\infty$, denote the class of measurable functions of period 2π for which $(\int_{-\pi}^{\pi} |f(x)|^p dx)^{1/p} = M_p(f) < \infty$, and let Lip (α, p), $0<\alpha<\infty$, represent that subclass of L_p for which $(\int_{-\pi}^{\pi} |f(x+h) - f(x)|^p dx)^{1/p} = O(h^{-\alpha})$ as $h \to 0$. The object of the present note is to demonstrate the following theorem.

THEOREM. *If $f(x) \in$ Lip (α, p) and $\{P_n(x)\}$ is a sequence of trigonometric polynomials of order n such that*

$$M_p(f - P_n) \leqq Kn^{-\alpha}, \tag{1}$$

then

$$\left(\int_{-\pi}^{\pi} |P_n'(x)|^p dx\right)^{1/p} \leqq \begin{cases} A(1-\alpha)^{-1}n^{1-\alpha}, & 0<\alpha<1, \\ A \log n, & \alpha = 1, \\ A(\alpha-1)^{-1}, & 1<\alpha<\infty \end{cases} \tag{2}$$

where in each case A depends only on α and the sequence $P_n(x)$ but not on n.

The method is that of M. Zamansky[1] [2] who obtained the corresponding results for functions in Lip α, $0<\alpha\leqq 1$.

An application of the inequality of Zygmund [3] concerning the pth mean of the derivative of a trigonometric polynomial together with the Minkowski inequality shows that if (1) and (2) are satisfied by a sequence $\{P_{n_j}\}$ with $(n_{j+1}/n_j) = O(1)$ and if $\{\lambda_n\}$ is any sequence of trigonometric polynomials of order n such that $M_p(\lambda_n) = O(n^{-\alpha})$, then the sequence $\{P_{n_j} + \lambda_n\}$ $(n = n_j, n_j+1, \cdots, n_{j+1}-1; j = 1, 2, \cdots)$ also satisfies (1) and (2). A further application of the same inequalities shows that if $\{P_n\}$ satisfies (1) and (2) and if $\{Q_n\}$ satisfies (1), then $\{Q_n\}$ also satisfies (2). The proof of the theorem is thus reduced to the exhibition of a sequence $\{P_{n_j}\}$ of trigonometric polynomials of order n_j with $(n_{j+1}/n_j) = O(1)$ such that (1) and (2) hold for $\{P_{n_j}\}$.

Let r be the smallest integer greater than $(1+\alpha)/2$ and $q = p/(p-1)$. If $f(x) \in L_p$ and

$$u(r) = \int_{-\infty}^{\infty} (\sin t/t)^{2r} dt$$

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[1] Numbers in brackets refer to the references at the end of the note.