ABSOLUTE-VALUED ALGEBRAIC ALGEBRAS

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1. **Introduction.** An *algebra* $\mathfrak{A}$ over a field $\mathfrak{F}$ is a vector space over $\mathfrak{F}$ which is closed with respect to a product xy which is linear in both x and y. The product is not necessarily associative. Every element x of $\mathfrak{A}$ generates a subalgebra $\mathfrak{F}[x]$ of $\mathfrak{A}$ and we call $\mathfrak{A}$ an algebraic algebra if every $\mathfrak{F}[x]$ is a finite-dimensional vector space over $\mathfrak{F}$.

We have shown elsewhere[1] that every absolute-valued real finite-dimensional algebra has dimension 1, 2, 4, or 8 and is either the field $\mathfrak{R}$ of all real numbers, the complex field $\mathfrak{C}$, the real quaternion algebra $\mathfrak{Q}$, the real Cayley algebra $\mathfrak{D}$, or certain isotopes without unity quantities of $\mathfrak{Q}$ and $\mathfrak{D}$. In the present paper we shall extend these results to algebraic algebras over $\mathfrak{R}$ showing that every algebraic algebra over $\mathfrak{R}$ with a unity quantity is finite-dimensional and so is one of the algebras listed above. The results are extended immediately to absolute-valued algebraic division algebras, that is, to algebras without unity quantities whose nonzero quantities form a quasigroup.

2. **Quadratic algebras.** Let $\mathfrak{F}$ be a field whose characteristic is not two and $\mathfrak{A}$ be an algebra over $\mathfrak{F}$ with a unity quantity 1. The scalar multiples $\alpha 1$ defined for α in $\mathfrak{F}$ form a subalgebra of $\mathfrak{A}$ isomorphic to $\mathfrak{F}$ and we may assume that $\mathfrak{F}$ is actually a subalgebra of $\mathfrak{A}$ whose unity element coincides with that of $\mathfrak{A}$. Then $\mathfrak{F}$ is a subalgebra of the center[2] of $\mathfrak{A}$. We shall call the elements of $\mathfrak{A}$ which are in $\mathfrak{F}$ the *scalars* of $\mathfrak{A}$ and all other elements of $\mathfrak{A}$ *nonscalars.*

In an algebra of degree two over $\mathfrak{F}$ every x is a root of an equation

$$f(\xi, x) \equiv \xi^2 - 2\xi\tau(x) - \nu(x) = 0, \tag{1}$$

where $\tau(x)$ and $\nu(x)$ are functions on $\mathfrak{A}$ to $\mathfrak{F}$. Then every nonscalar x of $\mathfrak{A}$ determines a commutative associative algebra $\mathfrak{F}[x] = \mathfrak{F} + x\mathfrak{F}$ of order two over $\mathfrak{F}$. It is seen trivially that $f(\xi, x)$ is unique for every

Presented to the Society, September 10, 1948; received by the editors June 21, 1948.

[1] See the author's *Absolute valued real algebras*, Ann. of Math. vol. 48 (1947) pp. 495–501.

[2] The center of a finite-dimensional algebra was defined in the author's *Non-associative algebras* I, Ann. of Math. vol. 43 (1942) on page 707. The same definition was given later for rings by T. Nakayama, *Über einfache distributive Systeme unendlicher Range,* Proc. Imp. Acad. Tokyo vol. 20 (1944) p. 62 and by N. Jacobson, *Structure theory of rings without finiteness assumptions,* Trans. Amer. Math. Soc. vol. 57 (1945) p. 239.