ON THE EXISTENCE OF THE INVERSE OPERATION IN ALTERNATION GROUPOIDS[1]

MARLOW SHOLANDER

Introduction. An Abelian quasigroup S may be defined as a set of elements $a, b, c, \cdots$ for which the following postulates hold:

Postulate I. There is an equivalence relation in S, denoted by "$=$"—that is, equality of elements is reflexive, symmetric, and transitive. There is a binary operation in S, denoted by # or, when convenient, by the notation of multiplication. It is understood that this implies S is closed with respect to #, and that # is uniquely defined in S—that is, if $a=b$ and $c=d$, then $ac=bd$.

Postulate II. # is an alternation—that is, if a, b, c, and d are elements in S, $(ab)(cd)=(ac)(bd)$.

Postulate III. Each element in S is proper (see §2 for definitions).

D. C. Murdoch [1, p. 516][2] pointed out that an Abelian quasigroup is a natural generalization of an Abelian group. The same author [2, Theorem 11] proved that in an Abelian quasigroup it is always possible to define a new operation under which the elements of the quasigroup form an Abelian group. R. H. Bruck [3, Theorem 12] in a sense completed the theory by showing how all Abelian quasigroups may be derived from Abelian groups.

In this paper a set of elements S for which postulates I and II hold is called an alternation groupoid. A series of extensions of an alternation groupoid is described which for certain groupoids leads to an Abelian quasigroup (Corollary 5.5). This imbedding process has as a special case the well known procedure for imbedding commutative semigroups in groups or, more generally, the procedure for imbedding a space S, in which there is a commutative and associative operation, in a space S' in such a way that each "regular" element in S has an inverse in S' (see, for example, [4, p. 24]). It should be noted that A. Malcev [5, §2] has given an example of a noncommutative semigroup which cannot be imbedded in a group.

Some interesting examples of Abelian quasigroups and alternation groupoids are given in §1.

1. Examples of alternation groupoids. The examples of Abelian

Presented to the Society, November 30, 1946; received by the editors June 12, 1948.

[1] A part of this paper was written while the author was receiving support from the Office of Naval Research.

[2] Numbers in brackets refer to references cited at the end of the paper.