SPLITTING CRITERIA AND EXTENSION TYPES[1]

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Introduction. Following the lead given by Galois theory it appears natural to term extensions G and H of the group N equivalent extensions, if there exists an isomorphism of G upon H which leaves invariant every element in N. But if G is an extension of N, and if S is some group, then it will, in general, be impossible to assert the equivalence of the extensions G and $G\oplus S$ of N in spite of the fact that these two extensions of N do not differ in any interesting feature. Thus one may feel that the concept of equivalence provides too narrow a principle of classification of extensions. With this in mind we say that the extension type of the extension G of N is contained in the extension type of the extension H of N, if there exists a homomorphism of G into H which leaves invariant every element in N and which maps normal subgroups of G upon normal subgroups of H; and we say that the extensions G and H of N belong to the same extension type if the type of G is contained in the type of H and the type of H is contained in the type of G. In this way we obtain a partially ordered set of extension types of N (§3).

It is clear that the extensions G and $G\oplus S$ of N, mentioned before, belong to the same extension type of N; and more generally $G\oplus S$ and $G\oplus T$ belong to the same extension type of N, if G is an extension of N. In order to obtain criteria for the validity of the converse we need splitting criteria asserting the existence of complements which are normal subgroups. Since hardly any such criteria are available in the literature (see Baer [2][2] for references) we devote the first section to the derivation of such criteria which may be of interest independent of the applications in §3. In these applications there exists always a normal subgroup whose elements are left invariant by the endomorphisms under consideration. This makes it possible to weaken the hypotheses usually needed; and a systematic account of these phenomena is given in §2. In an appendix we show that finitely generated groups are not isomorphic to proper quotient groups, if their normality preserving endomorphisms split.

1. Normality and uniformity of splitting. We consider M-groups G where the composition of elements in G is addition $x+y$ and where

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[2] Numbers in brackets refer to the bibliography at the end of the paper.