PROBABILITY METHODS IN SOME PROBLEMS OF ANALYSIS AND NUMBER THEORY

M. KAC

1. **Introduction.** In 1922 Rademacher [1][1] introduced the functions

$$r_n(t) = \text{sign} \sin 2^n \pi t, \qquad 0 \leqq t \leqq 1,\ n = 1, 2, \cdots, \tag{1.1}$$

and proved that the series

$$\sum_1^\infty c_n r_n(t) \tag{1.2}$$

converges almost everywhere provided

$$\sum_1^\infty c_n^2 < \infty. \tag{1.3}$$

In 1925 Kolmogoroff and Khintchine [1] generalized this result and also proved the counterpart to the effect that

$$\sum_1^\infty c_n^2 = \infty \tag{1.4}$$

implies divergence almost everywhere of (1.2). The probabilistic nature of these results (first recognized by Steinhaus [1]) becomes apparent when one notices that the Rademacher functions $r_n(t)$ are *statistically independent*, that is, have the property that

$$\left| \underset{t}{E} \{r_1(t) < \alpha_1, \cdots, r_n(t) < \alpha_n\} \right| = \prod_1^n \left| \underset{t}{E} \{r_k(t) < \alpha_k\} \right|,^2 \tag{1.5}$$

for $n = 2, 3, \cdots$ and all real $\alpha_1, \alpha_2, \cdots$.

Following the natural line of development, Kolmogoroff [1; 2] was led to his celebrated necessary and sufficient conditions (the "three series theorem") for convergence of series,

$$\sum_1^\infty f_n(t), \tag{1.6}$$

An address delivered before the Annual Meeting of the Society in Columbus, Ohio, on December 28, 1948, by invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings; received by the editors November 1, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Here, as in the sequel, $E\{\ \}$ denotes the set of t's satisfying the condition inside the braces, and $|A|$ denotes the Lebesgue measure of the set A.