THE NORMAL APPROXIMATION TO THE POISSON DISTRIBUTION AND A PROOF OF A CONJECTURE OF RAMANUJAN[1]

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1. Summary. The Poisson distribution with parameter λ is given by

$$(1.1)\qquad F(x) = \sum_{r=0}^{n} p_r \quad \text{where} \quad p_r = \frac{\lambda^r}{r!} e^{-\lambda}, \qquad n = [x],$$

for $x \geqq 0$. It is well known that $F(x)$ converges to the normal distribution as $\lambda \to \infty$. We shall prove the following theorem.

THEOREM I. *Let* $x = \lambda^{-1/2}(n - \lambda + 1/2)$. *Then*

$$(1.2)\qquad \begin{aligned} \sum_{r=0}^{n} \frac{\lambda^r}{r!} e^{-\lambda} = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-(1/2)t^2} dt \\ + (1/6)(2\pi\lambda)^{-1/2}(1 - x^2)e^{-(1/2)x^2} + \delta, \end{aligned}$$

where δ *satisfies the inequality*

$$(1.3)\qquad |\delta| < .076\lambda^{-1} + .043\lambda^{-3/2} + .13\lambda^{-2}.$$

This formula is analogous to Uspensky's [1][2] estimate of the error term in the normal approximation to the binormal distribution and our proof consists in an adaptation of Uspensky's method.

At the same time we verify the following conjecture of Ramanujan: *for every positive integer* n *the value of* θ *which satisfies the equation*

$$(1.4)\qquad \left\{1 + \frac{n}{1!} + \cdots + \frac{n^{n-1}}{(n-1)!} + \theta \frac{n^n}{n!}\right\} e^{-n} = 1/2$$

lies between $1/2$ *and* $1/3$, *and tends to* $1/3$ *as* $n \to \infty$. A proof of the above statement was given by Szegö [3]. We give a more elementary proof, using only standard tools, for the following theorem.

THEOREM II. *For* $n \geqq 7$ *the root of the equation* (1.4) *in* θ *lies between* $.37$ *and* $1/3$.

Finally, we shall obtain the following asymptotic expansion of θ:

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[2] Numbers in brackets refer to the references cited at the end of the paper.