ORTHOGONALITY PROPERTIES OF C-FRACTIONS

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1. **Introduction.** It has been indicated in the work of Tchebichef and Stieltjes that the denominators $D_p(z)$ of the successive approximants of a J-fraction

$$\frac{b_0}{d_1 + z} - \frac{b_1}{d_2 + z} - \frac{b_2}{d_3 + z} - \cdots \tag{1.1}$$

constitute a sequence of orthogonal polynomials. The orthogonality relations which exist between the $D_p(z)$ may be expressed in the following way (cf. [4, 7]).[1] Let S' be defined as the operator which replaces every z^p by c_p in any polynomial upon which it operates, where the $\{c_p\}$ are a given sequence of constants. Then the orthogonality relations

$$S'(D_p(z)D_q(z))\begin{cases} = 0 & \text{for} \quad p \neq q, \\ \neq 0 & \text{for} \quad p = q, \end{cases} \tag{1.2}$$

hold relative to the operator S' and the sequence $\{c_p\}$. The polynomials $D_p(z)$ are given recurrently by the formulas $D_0(z)=1$, $D_p(z)=(d_p+z)D_{p-1}(z)-b_{p-1}D_{p-2}(z)$, $p=1, 2, \cdots$ $(D_{-1}(z)=0)$.

In this paper orthogonality relations similar to (1.2) are developed for the polynomials $B_p^*(z)$ which are derived from the denominators $B_p(z)$ of the successive approximants of a C-fraction

$$1 + \frac{a_1z^{\alpha_1}}{1} + \frac{a_2z^{\alpha_2}}{1} + \frac{a_3z^{\alpha_3}}{1} + \cdots, \tag{1.3}$$

where the a_p denote complex numbers and the α_p positive integers (cf. [3]). In fact, conditions (1.2) for a certain J-fraction are shown to be a specialization of the orthogonality relations for a C-fraction. Furthermore, necessary and sufficient conditions are obtained for the unique existence of the polynomials $B_p^*(z)$ in terms of the sequence $\{c_p\}$ (Theorem 2.2).

2. **Orthogonal polynomials constructed from the denominators of the approximants of a C-fraction.** Let $A_p(z)$ and $B_p(z)$ denote the numerator and denominator, respectively, of the pth approximant

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[1] Numbers in brackets refer to the bibliography at the end of the paper.