A NOTE ON THE ERGODIC THEOREMS

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Introduction, definitions and remarks. The purpose of this note is to give an example of a measurable transformation of a measure space onto itself for which the individual ergodic theorem holds while the mean ergodic theorem does not hold.

Let S be a measure space of finite measure, m the measure defined on the measurable subsets of S, and T a 1-1 point transformation of S onto itself which is measurable (both T and T^{-1} transform measurable sets into measurable sets). Let the points of S be denoted by y and let $f(y)$ be any real valued function defined on S. We denote by $F_h(y)$ the average $(1/h) \sum_{i=0}^{h-1} f(T^i y)$.

We shall say that the individual ergodic theorem holds for $f(y)$ if the sequence of averages $\{F_h(y)\}$ converges to a finite limit almost everywhere. If the individual ergodic theorem holds for every integrable function $f \in L_1(m)$ we shall say that the individual ergodic theorem holds (with respect to m).[1]

We shall say that the mean ergodic theorem holds in $L_p(m)$ $(p \geqq 1)$ for a function $f \in L_p(m)$ if $F_h(y) \in L_p(m)$ for $h=1, 2, \cdots$ and the sequence $\{F_h(y)\}$ converges in the norm of $L_p(m)$. If the mean ergodic theorem holds in $L_p(m)$ for every function $f(y) \in L_p(m)$ then we shall say that the mean ergodic theorem holds in $L_p(m)$.

The following relations between the two ergodic theorems are known: If T is measure preserving, both the individual [**1**] and the mean [**4**][2] ergodic theorems hold. Without assuming that T is measure preserving, the mean ergodic theorem in $L_p(m)$ for any $p \geqq 1$ implies the individual ergodic theorem for all functions in $L_p(m)$ ([**2**, p. 1061], see also [**3**, p. 539] for the case $p=1$).

The question arises whether, conversely, the individual ergodic theorem implies the mean ergodic theorem in $b_p(m)$ for some $p \geqq 1$. This question has significance only when $L_p(m)$ is transformed into itself by the transformation induced on it by T. For in this case and only in this case is it true that for any $f \in L_p(m)$ the averages $\{F_h\}$ also belong to $L_p(m)$ for $h=1, 2, 3, \cdots$.[3] We answer this question in

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[1] The words in the parenthesis will be omitted if there is no reason for ambiguity.

[2] Only the case $p=2$ is proved in [**4**]; see [**2**, p. 1053] for all $p \geqq 1$. The numbers in brackets refer to the bibliography at the end of the paper.

[3] It is easy to give examples for which the individual ergodic theorem does hold while $L_p(m)$ is not transformed into itself. Such an example for instance is given if T is periodic while m is non-atomic and T^{-1} is singular.