ON THE SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER INVARIANT UNDER CONTACT TRANSFORMATIONS

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In the classical Lie theory it is shown how to construct a differential equation invariant under a given group, and how to solve an equation when a group leaving the equation invariant is known. However, little is said about the problem of determining the group for a given differential equation, which is by far the most interesting problem.

In the present paper, necessary and sufficient conditions for the existence of an infinitesimal contact transformation leaving a given equation invariant are determined along with the general form of the characteristic function of the group. It will also be shown how to reduce, by a proper change of variables, the infinitesimal contact transformation to a point transformation. This enables one to solve the transformed differential equation by Lie's methods. Passing back to the original variables, a new differential equation is obtained which combined with the original equation gives its solution in parametric form.

Let

$$Bf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \pi \frac{\partial f}{\partial p}$$

be the symbol of the infinitesimal contact transformation leaving invariant the differential equation $u = F(v)$, with $u = u(x, y, p)$, $v = v(x, y, p)$, $p = dy/dx$, and F such that the equation $G(x, y, p) = u - F(v) = 0$ satisfies the various conditions for the existence of solutions (but otherwise arbitrary). Throughout this paper we shall assume that:

(A) Both u and v have first derivatives with respect to x, y and p, at least in some region R of the (x, y, p)-space.

(B) The Jacobians

$$J_1 = \frac{\partial(u, v)}{\partial(y, p)}, \qquad J_2 = \frac{\partial(u, v)}{\partial(p, x)}, \qquad J_3 = \frac{\partial(u, v)}{\partial(x, y)}$$

have in R derivatives of the first and second orders, while J_1 and J_2 have also derivatives of the third order with respect to x, y and p,

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