COMBINATORIAL HOMOTOPY. I

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1. Introduction. This is the first of a series of papers, whose aim is to clarify the theory of "nuclei" and "n-groups" and its relation to Reidemeister's[1] *Überlagerungen*. Here we give a new definition of "n-groups," or *n-types* as we now propose to call them. This is stated in terms of $(n-1)$-homotopy types, which were introduced by R. H. Fox.[2] In a later paper we shall show that this is equivalent to the definition in terms of elementary transformations, which was given in [1]. The series of n-types ($n=1, 2, \cdots$) is a hierarchy of homotopy, and a fortiori of topological invariants. That is to say, if two complexes,[3] K, L, are of the same n-type, then they are of the same m-type for any $m<n$, where $n \leqq \infty$ and the ∞-type means the homotopy type. If dim K, dim $L \leqq n$ then K, L are of the same homotopy type if they are of the same $(n+1)$-type. Two complexes are of the same 2-type if, and only if, their fundamental groups are isomorphic. Moreover any (discrete) group is isomorphic to the fundamental group of a suitably constructed complex. Therefore the classification of complexes according to their 2-types is equivalent to the classification of groups by the relation of isomorphism. Thus the n-type ($n>2$) is a natural generalization of a geometrical equivalent of an abstract group.[4]

Following up this idea we look for a purely algebraic equivalent of an n-type when $n>2$. An important requirement for such an algebraic system is "realizability," in two senses. In the first instance this means that there is a complex which is in the appropriate relation to a given one of these algebraic systems, just as there is a complex whose fundamental group is isomorphic to a given group. The second kind, whose importance is underlined by theorems in [5; 6] and in this paper, is the "realizability" of homomorphisms, chain mappings, etc., by maps of the corresponding complexes. Thus realizability means that the algebraic representation is not subject to conditions which can only be expressed geometrically.

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1 See [1], [3] and [8, p. 177]. Numbers in brackets refer to the references cited at the end of the paper.

2 See [9, p. 343] and [10, p. 49].

3 I.e., CW-complexes, as defined in §5 below.

4 I.e., the class of groups which are isomorphic to a given group.