A PROPERTY OF POWER SERIES WITH POSITIVE COEFFICIENTS

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The following theorem is suggested by a problem in the theory of probability.[1]

Let p_k *be a sequence of non-negative numbers for which* $\sum_0^\infty p_k = 1$, *and let* $m = \sum_1^\infty k p_k \leqq \infty$. *Suppose further that*

$$P(x) = \sum_0^\infty p_k x^k$$

is not a power series in x^t *for any integer* $t > 1$. *Then* $1 - P(x)$ *has no zeros in the circle* $|x| < 1$, *and the series*

$$U(x) = \frac{1}{1 - P(x)} = \sum_0^\infty u_k x^k$$

has the property

$$\lim_{n \to \infty} u_n = 1/m.$$

(*If* $m = \infty$, *we define* $1/m$ *to be zero.*)

We shall first give a proof in case $m < \infty$. The method used is not elementary, but yields somewhat more information than stated in the theorem. Later in this paper an elementary proof is given, valid for both $m < \infty$ and $m = \infty$.

We suppose that $m < \infty$. Let

$$(1) \qquad r_n = \sum_{k=n+1}^\infty p_k, \qquad R(x) = \sum_0^\infty r_n x^n.$$

Then $m = \sum_0^\infty r_k$ and

$$(2) \qquad 1 - P(x) = (1 - x) R(x).$$

Since $m < \infty$ the power series for $R(x)$ converges absolutely and uniformly in $|x| \leqq 1$. We claim that $R(x)$ has no zeros for $|x| \leqq 1$. For $|x| < 1$ this is clear from (2), since $P(x)$ has positive coefficients

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