A METHOD OF ANALYTIC CONTINUATION SUGGESTED BY HEURISTIC PRINCIPLES

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Suppose we are given an analytic function $f(z)$ represented by a power series (supposed convergent for some $z \neq 0$)

$$f(z) = \sum_{n=0}^{\infty} \frac{f_n}{n!} z^n \tag{1}$$

where f_n is the value of the nth derivative of $f(z)$ at the origin. For small values of δ we can approximate $f(\delta)$ in the following manner:

$$f(\delta) \simeq f_0 + \delta f_1. \tag{2}$$

Refinement of this approximation leads to Taylor's theorem and back to the power series (1). It is possible, however, to use the linear approximation in a different way: we can use such approximations to go from one point to another along a chain of points $z = \delta$, 2δ, $3\delta, \cdots, n\delta$. Thus we shall say

$$f(\delta) \simeq f_0 + \delta f_1, \qquad f'(z)\big|_{z=\delta} \simeq f_1 + \delta f_2,$$

and so on, and

$$f(2\delta) \simeq f(\delta) + \delta f'(z)\big|_{z=\delta} \simeq f_0 + 2\delta f_1 + \delta^2 f_2.$$

In general

$$f(n\delta) \simeq \sum_{m=0}^{n} a_{n,m} f_m \delta^m. \tag{3}$$

It is easily verified that $a_{n,m} = a_{n-1,m} + a_{n-1,m-1}$ and hence that $a_{n,m}$ is the binomial coefficient $C_{n,m}$. If we now define the following:

$$\sigma_n(z) = \sum_{m=0}^{n} f_m C_{n,m} \left(\frac{z}{n}\right)^m \tag{4}$$

and if $n\delta = z$, then (3) is equivalent to

$$f(z) \simeq \sigma_n(z). \tag{5}$$

The question now presents itself: for what values of z does the sequence of polynomials $\sigma_n(z)$ converge to the function $f(z)$? It is evident that if z is inside the circle of convergence of the series (1), then

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