ON THE EXTENSION OF A TRANSFORMATION

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0. Introduction. In a problem on surface area the writer and Helsel[1] were confronted with the following question. Can a Lipschitzian transformation from a set in a Euclidean three-space into a Euclidean three-space be extended to a Lipschitzian transformation defined on the whole space? The affirmative answer to this question has been given by Kirszbraun.[2] In fact, Kirszbraun shows this result for any Euclidean spaces (see also Valentine).[3] In studying these papers the writer noted that a more general extension problem could be formulated and a different method of proof to the problem could be obtained. To formulate the more general problem we first give some definitions.

Let M be a metric space, the distance between two points $p_1, p_2 \in M$ being denoted by p_1p_2. Let $\mathfrak{P}(M)$ be the class of real-valued continuous functions $g(t)$, $0 \leqq t < \infty$, which satisfy the conditions: (a) $g(0)=0$, (b) $g(t)>0$ for $t>0$, (c) for any finite number of points $p_0, p_1, \cdots, p_m$ in M the real quadratic form $\sum_{i,j=1}^{m} [g(p_0p_i)^2+g(p_0p_j)^2 - g(p_ip_j)^2]\xi_i\xi_j$ is positive. Let $g(t) \in \mathfrak{P}(M)$. A transformation $p^*=\phi(p)$ from a set E in M into a metric space M^* will be said to satisfy the condition $C(g)$ on E if, for every pair of points $p_1, p_2 \in E$, $p_1^*p_2^* \leqq g(p_1p_2)$, where $p_i^*=\phi(p_i)$, $i=1, 2$. We shall say that $\phi(p)$ can be extended to a set E', $E \subset E' \subset M$, preserving the condition $C(g)$ if there exists a transformation $p^*=\Phi(p)$ from E' into M^* which satisfies the condition $C(g)$ on E' and is equal to $\phi(p)$ on E.

In this paper we prove the following result. Let M be a separable metric space and let $g(t) \in \mathfrak{P}(M)$. Then any transformation from a set E in M into a Euclidean space which satisfies the condition $C(g)$ on E can be extended to M preserving the condition $C(g)$.

We give two examples to illustrate this result. We shall use the vector notation x to represent a point in a Euclidean n-space E_n, and we shall denote by $|x_1-x_2|$ the distance between two points x_1, x_2. Let $x_0, x_1, \cdots, x_m$ be $m+1$ points in E_n and let $\xi_1, \cdots, \xi_m$ be m real numbers. From the relation $(x_i-x_j)^2=(x_0-x_i)^2+(x_0-x_j)^2 -2(x_0-x_i)(x_0-x_j)$, the square of the vector $x=L\xi_1(x_0-x_1)+ \cdots$

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[1] Helsel and Mickle, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 235–238.

[2] Kirszbraun, Fund. Math. vol. 22 (1934) pp. 77–108.

[3] Valentine, Bull Amer. Math. Soc. vol. 49 (1943) pp. 100–108.