COVERINGS AND BETTI NUMBERS

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1. **Introduction.** Let the finite polyhedron P be a regular covering of the polyhedron $\bar{P}$, and let G be the corresponding (finite) group of covering transformations of P. Then G acts as operator group on the homology groups H_n of P. If we consider homology with real coefficients, H_n is a real vector group of finite rank $p_n = n$th Betti number of P, and G operates in H_n as a group of linear transformations; we denote by $s_n(x)$, $x \in G$, the character of this linear representation of G of degree p_n. Let g be the order of G.

In this note we shall prove:

THEOREM 1. *The nth Betti number* $\bar{p}_n$ *of* $\bar{P}$ *is given by*

$$\bar{p}_n = \frac{1}{g} \sum_{x \in G} s_n(x).$$

In the Princeton Bicentennial Conference W. Hurewicz raised the question whether the homology groups of a polyhedron $\bar{P}$ are determined by those of a regular covering P given as groups with operators. According to Theorem 1 the answer is affirmative in the case of *finite* polyhedra P, $\bar{P}$ and real (or rational) coefficients. We shall show elsewhere—in a general theory of complexes with automorphisms—that the same is true for arbitrary (finite or infinite) polyhedra, provided that the group G is finite. Since the proof in our present case, based upon "harmonic chains," is very simple and yields the explicit formula of Theorem 1, which has interesting applications, we give it here independently of other more general considerations.

2. **Simplicial covering and harmonic chains.** To compute the homology groups of P and $\bar{P}$, let K and $\bar{K}$ be finite simplicial complexes which are subdivisions of P and $\bar{P}$ respectively, such that each oriented simplex of K covers one oriented simplex of $\bar{K}$. Then G acts as an *automorphism group*[1] on K; the automorphisms $x \in G$ are permutations of the oriented simplices σ_n of K in each dimension n and preserve all incidence relations. The set of all simplices σ_n covering a simplex $\bar{\sigma}_n$ of $\bar{K}$ is a transitivity domain of G; that is, it contains with any simplex σ_n' all $x\sigma_n'$, $x \in G$, and only those. Furthermore, since in an automorphism $x \neq e$ no simplex is fixed,

Presented to the Society, April 17, 1948; received by the editors February 25, 1948.

[1] For details see for example [1, §6]. Numbers in brackets refer to the references at the end of the paper.