THE DIFFERENTIAL INVARIANTS OF A TWO-INDEX TENSOR

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Riemannian geometry, based upon a metric form $ds^2 = g_{ij}dx^i dx^j$, gives us the curvature tensor R^i_{jkl} as the sole basic differential invariant of the space, and of the symmetric tensor g_{ij}. The general tensor g_{ij} can be broken up into the sum of two irreducible components, namely the symmetric and antisymmetric portions defined respectively by $2g_{(ij)} = g_{ij} + g_{ji}$ and $2g_{[ij]} = g_{ij} - g_{ji}$. The latter disappears in constructing ds^2; but the general differential invariants of g_{ij} must necessarily be composed of those derivable from $g_{(ij)}$ (the curvature tensor above), from $g_{[ij]}$, and a group of mixed invariants dependent upon both. It is proposed to investigate the general problem by use of a well known and easily proved fundamental lemma of the calculus of variations: *The Euler equations derived from a variational principle are tensor-invariant under the group of transformations which leaves the original integral invariant.* Actually the equations as directly obtained state that a certain covariant vector vanishes.

Given the tensor $g_{ij}(x^1 \cdots x^n)$ we first introduce two (implicit) absolute parameters u, v, and construct the variational problem

$$(1) \qquad \delta \int g_{ij} \overset{i}{x}_u \overset{j}{x}_v du dv = 0; \qquad \overset{i}{x}_u = \frac{\partial x^i}{\partial u}, \quad \text{and so on.}$$

Only x-transformations will be allowed for the present. The Euler equations become

$$(2) \qquad \begin{gathered} 2\{ g_{(ij)} \overset{j}{x}_{uv} + L_{jki} \overset{j}{x}_u \overset{k}{x}_v \} = 0; \\ L_{jki} = (g_{ik,j} + g_{ji,k} - g_{jk,i})/2; \qquad g_{ij,k} = \partial g_{ij}/\partial x^k. \end{gathered}$$

These L_{ijk} must, therefore, have the law of transformation of Christoffel symbols of the first kind. In fact

$$(3) \qquad \begin{aligned} L_{jki} &= \{ g_{(ik),j} + g_{(ij),k} - g_{(jk),i} \}/2 + \{ g_{[ik],j} + g_{[ji],k} + g_{[kj],i} \}/2 \\ &= \Gamma_{jki} + \Omega_{jki}. \end{aligned}$$

Here Γ_{jki} are precisely Christoffel symbols of the first kind associated with $g_{(ij)}$, and Ω_{jki} *is the fully covariant form of the Cartan torsion tensor.* If now the discriminant $|g_{(ij)}| \neq 0$, we may construct $g^{(ij)}$ as usual to raise indices, and then obtain the coefficients of a general

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