A CHARACTERIZATION OF CONFORMALLY FLAT SPACES

A. H. TAUB[1]

1. Introduction. The purpose of this paper is to show some analogies between the theory of groups of infinitesimal conformal transformations and that of groups of motions in Riemannian spaces. The main part of the paper will be devoted to Theorem I: *The necessary and sufficient condition for an n-dimensional* ($n \geqq 3$) *Riemannian space,* V_n, *to admit an* $(n+1)(n+2)/2$ *parameter group of infinitesimal conformal transformations is that the* V_n *be conformally flat.*

This theorem is the analogue of the well known theorem concerning groups of motions of n-dimensional Riemannian spaces V_n, namely,[2] *A group of motions of a* V_n *has at most* $n(n+1)/2$ *parameters and this number only in case* V_n *has constant curvature.* Sasaki[3] has shown part of Theorem I by using the formalism of the conformal connection. He has shown that a space with a conformal connection will admit a maximal number of independent infinitesimal conformal transformations only if it is conformally flat but has not given this number. The proof given here of Theorem I will be carried out in a fashion similar to Eisenhart's proof of the latter one. We shall use the notation of R. G. throughout this paper.

In a V_n ($n>3$) for which the Weyl conformal curvature tensor vanishes, there exists a coordinate system[4] where

$$g_{ij} = e^{2\sigma(x)} e_i \delta_{ij}, \qquad e_i = \pm 1. \tag{1.1}$$

That is, the V_n is conformal to a flat space S_n with the metric

$$a_{ij} = e_i \delta_{ij}.$$

The S_n admits[5] the conformal group of $(n+1)(n+2)/2$ parameters generated by dilatation, inversions, translations and orthogonal transformations. The part of this group connected to the identity is

Received by the editors February 21, 1948.

[1] Guggenheim post-Service Fellow at the Institute for Advanced Study, on leave from the University of Washington.

[2] L. P. Eisenhart, *Riemannian geometry*, Princeton University Press, 1926, p. 239. This work will be referred to as R. G. hereafter.

[3] Shigeo Sasaki, *Geometry of the conformal connexion*, Science Reports of the Tôhoku Imperial University (1) vol. 29 (1940) pp. 219–267. I am indebted to J. L. Vanderslice for this reference and a valuable suggestion regarding the integrability condition of equation (3.5).

[4] R. G. p. 92.

[5] R. G. p. 94, problem 16.