PARACOMPACTNESS AND PRODUCT SPACES

A. H. STONE

A topological space is called *paracompact* (see [2])[1] if (i) it is a Hausdorff space (satisfying the T_2 axiom of [1]), and (ii) every open covering of it can be refined by one which is "locally finite" (=neighbourhood-finite; that is, every point of the space has a neighbourhood meeting only a finite number of sets of the refining covering). J. Dieudonné has proved [2, Theorem 4] that every *separable* metric (=metrisable) space is paracompact, and has conjectured that this remains true without separability. We shall show that this is indeed the case. In fact, more is true: paracompactness is identical with the property of *"full normality"* introduced by J. W. Tukey [5, p. 53]. After proving this (Theorems 1 and 2 below) we apply Theorem 1 to obtain a necessary and sufficient condition for the topological product of uncountably many metric spaces to be normal (Theorem 4).

For any open covering $\mathcal{W} = \{W_\alpha\}$ of a topological space, the *star* $(x, \mathcal{W})$ of a point x is defined to be the union of all the sets W_α which contain x. The space is *fully normal* if every open covering $\mathcal{V}$ of it has a "*Δ-refinement*" $\mathcal{W}$—that is, an open covering for which the stars $(x, \mathcal{W})$ form a covering which refines $\mathcal{V}$.

THEOREM 1. *Every fully normal T_1 space is paracompact.*

Let S be such a space, and let $\mathcal{V} = \{U_\alpha\}$ be a given open covering of S. (We must construct a locally finite refinement of $\mathcal{V}$. Note that S is normal [5, p. 49] and thus satisfies the T_2 axiom.)

There exists an open covering $\mathcal{V}^1 = \{U^1\}$ which Δ-refines $\mathcal{V}$, and by induction we obtain open coverings $\mathcal{V}^n = \{U^n\}$ of S such that $\mathcal{V}^{n+1}$ Δ-refines $\mathcal{V}^n$ ($n = 1, 2, \cdots$, to ∞). For brevity we write, for any $X \subset S$,

$$(1) \qquad \begin{aligned} (X, n) &= \text{star of } X \text{ in } \mathcal{V}^n \\ &= \text{union of all sets } U^n \text{ meeting } X \end{aligned}$$

(roughly corresponding to the "$1/2^n$-neighbourhood of X" in a metric space), and

$$(2) \qquad (X, -n) = S - (S - X, n).$$

Presented to the Society, October 25, 1947; received by the editors October 25, 1947.

[1] Numbers in brackets refer to the bibliography at the end of the paper.