PROOF OF A THEOREM OF SAKS AND SIERPINSKI

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The object of this note is to give a direct proof of the following theorem of Saks and Sierpinski:

If $f(x)$ is an arbitrary one-valued real function defined on the closed interval $I=[0, 1]$, there is a $\phi(x)$ of Baire class 2 at most such that for every $\epsilon>0$ the inequality $|f(x)-\phi(x)|<\epsilon$ holds on a set of exterior measure 1.

The proofs in the literature [1, 2][1] depend on the corresponding theorem for measurable functions; namely, if $f(x)$ is measurable, there is a $\phi(x)$ of Baire class 2 at most such that $f(x)=\phi(x)$ almost everywhere.

We first prove a lemma which seems to be new.

LEMMA. *If $f(x)$ is defined on the closed interval $I=[0, 1]$, $\epsilon>0$, $\phi(x)$ continuous, and $|f(x)-\phi(x)|<\epsilon$ on a set of exterior measure greater than $1-\epsilon$, then for every $\eta>0$ there is a continuous $\psi(x)$ such that $|\phi(x)-\psi(x)|<\epsilon$ on a set of measure greater than $1-\epsilon$ and $|f(x)-\psi(x)|<\eta$ on a set of exterior measure greater than $1-\eta$.*

PROOF. $f(x)$ is exteriorly approximately continuous almost everywhere in the sense that for almost every $\xi\in I$ the set of points x for which $f(\xi)-k<f(x)<f(\xi)+k$ has exterior metric density 1 at ξ, for every $k>0$. There is a δ, with $0<\delta<\eta$, such that $|f(x)-\phi(x)|<\epsilon-\delta$ on a set E of exterior measure greater than $1-\epsilon$. Since $f(x)$ is exteriorly approximately continuous almost everywhere, every $\xi\in E-Z$, with Z of measure zero, is in a sequence, $\{I_{\xi n}\}$, $i=1, 2, \cdots$, of closed intervals, whose lengths converge to zero, such that the set of points x for which $|f(\xi)-f(x)<\delta$ has relative exterior measure exceeding $1-\delta/2$ in each $I_{\xi n}$. Moreover, since $\phi(x)$ is continuous, the $I_{\xi n}$ may be chosen so that the saltus of $\phi(x)$ in $I_{\xi n}$ is less than δ for every n. Consider the totality of intervals

$$I=[I_{\xi n}], \quad \xi\in E-Z, n=1, 2, \cdots .$$

By Vitali's covering theorem, since $m_e(E-Z)>1-\epsilon$, there is a finite number $I_{\xi_1 n_1}, I_{\xi_2 n_2}, \cdots, I_{\xi_k n_k}$ of disjoint intervals of I the sum, $\sum_{i=1}^k m(I_{\xi_i n_i})$, of whose lengths is $1-\alpha>1-\epsilon$. Let $G=I-\sum_{i=1}^k I_{\xi_i n_i}$ and let $G'\subset G$ be the points of G at which $f(x)$ is exteriorly approxi-

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[1] Numbers in brackets refer to the references cited at the end of the paper.