ON POLYNOMIALS AND LAGRANGE'S FORM OF THE GENERAL MEAN-VALUE THEOREM

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Suppose that in $(a<x<b)$ (hereafter referred to as (a, b)),

(1) $f(x)$ is defined and has derivatives of the first n orders.

Then, from the general mean-value theorem with Lagrange's form of remainder follows the existence of $\theta=\theta(x, h)$, such that

$$f(x+h) = f(x) + \sum_{r=1}^{n-1} \frac{h^r}{r!} f^{(r)}(x) + \frac{h^n}{n!} f^{(n)}(x+\theta h) \tag{2}$$

for $a < x < x+h < b$.

The θ in (2) is sometimes a uniquely determinate function of x and h in the relevant domain $a<x<x+h<b$ (hereafter referred to as R), as, for instance, if $f^{(n+1)}(x)$ exists and is not zero in (a, b). If, further, $f^{(n+1)}(x)$ is continuous in (a, b), it is easily seen that

$$\lim_{h\to+0} \theta(x, h) = \frac{1}{n+1} \qquad \text{in } a < x < b.$$

It is also possible for $\theta(x, h)$ to be an analytic function, for example,

$$\theta(x, h) = h^{-1} \log\left(1 + \sum_{r=1}^{\infty} \frac{h^r \Gamma(n+1)}{\Gamma(n+r+1)}\right),$$

which happens when $f(x)=e^x$.

It would, therefore, seem worth while to determine the types of functions that are or are not possible for $\theta(x, h)$. Inquiry in this direction has led to the results of this paper, namely:

THEOREM 1. *If a polynomial* $\theta(x, h)$ *exists such that* (2) *is true with* $\theta(x, h)$ *in place of* θ, *then* $f^{(n+1)}(x)$ *exists in* (a, b) *and either*

(a) $$f^{(n+1)}(x) = 0 \qquad \textit{in } (a, b)$$

or

(b) $f^{(n+1)}(x)=$ *a constant* $\neq 0$ *in* (a, b), *and* $\theta(x, h)$ *is uniquely determinate and equal to* $1/(n+1)$ *in* R.

THEOREM 2. *If* (2) *is true with* $\theta(x, h)=c(x)+h^d\phi(x, h)$ *where*

(3) $\phi(x, h)$ *is bounded in* R;

(4) d *is a constant greater than* 1;

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