ON THE LOCATION OF THE ZEROS OF THE DERIVATIVES OF A POLYNOMIAL SYMMETRIC IN THE ORIGIN

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If the zeros of a polynomial $p(z)$ when plotted in the z-plane are symmetric in 0: $z=0$, the zeros of the derivative $p'(z)$ of $p(z)$ can profitably be studied by transforming onto the w-plane, with $w=z^2$, and applying known theorems there.[1] It is the purpose of the present note to carry that study somewhat farther than has been previously done, in particular to consider the higher derivatives of $p(z)$.

Under the transformation $w=u+iv=z^2=(x+iy)^2$, an arbitrary line $Au+Bv+C=0$ in the w-plane corresponds to an equilateral hyperbola $A(x^2-y^2)+2Bxy+C=0$ in the z-plane with center 0 or to two perpendicular lines intersecting at 0. A half-plane in the w-plane for which $w=0$ is an interior or exterior point corresponds in the z-plane respectively to the exterior or interior of an equilateral hyperbola whose center is 0; a half-plane for which $w=0$ is a boundary point corresponds to a double sector with vertex $z=0$ and angle $\pi/2$. A point z is considered to be exterior or interior to a hyperbola according as the curve at its nearest point is convex or concave toward z.

We write the given polynomial in the form

$$p(z) = z^l \prod_{j=1}^{q} (z^2 - \alpha_j^2), \qquad \alpha_j \neq 0, \tag{1}$$

and in the w-plane study the polynomials ($w=z^2$)

$$P(w) = P(z^2) = [p(z)]^2, \qquad P'(w) = p(z)\cdot p'(z)/z. \tag{2}$$

Each zero of $P(w)$ corresponds to a zero of $p(z)$ and reciprocally; each zero of $P'(w)$ corresponds to a zero of $p(z)$ or $p'(z)$ and reciprocally except that $z=0$ is a zero of $p'(z)$ unless $z=0$ is a simple zero of $p(z)$.

We have (loc. cit.) by Lucas' Theorem

THEOREM 1. *If the zeros of $p(z)$ are symmetric in 0 and lie in the closed exterior of an equilateral hyperbola with center 0 or in the closed exterior of a double sector with vertex 0 and angle $\pi/2$, then the zeros of $p'(z)$ lie also in that closed exterior.*

If the zeros of $p(z)$ are symmetric in 0 and lie in the closed interior of an equilateral hyperbola with center 0, then the zeros of $p'(z)$ also lie in that closed interior except for a simple zero at 0.

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[1] Walsh, Mathematica vol. 8 (1933) pp. 185–190.