SOME ANALOGS OF THE GENERALIZED PRINCIPAL AXIS TRANSFORMATION

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It is known that two normal matrices can be diagonalized by the same unitary transformation if and only if they commute; this theorem is ordinarily stated for hermitian matrices. Some generalizations of this theorem are known. According to a theorem due to Eckert and Young,[1] if A and B are two $r\times s$ matrices, there are two unitary matrices U and V such that $UAV=D_1$ and $UBV=D_2$, D_1 and D_2 diagonal matrices with real elements, if and only if AB^{ct} and $B^{ct}A$ are hermitian. It is also known that a set of normal matrices $\{A_i\}$ is reducible to diagonal matrices under the same unitary similarity transformation, UA_iU^{ct}, if and only if $A_iA_j=A_jA_i$ for all i and j. (More generally, it is true that a set of matrices $\{A_i\}$ with elements in the complex field and simple elementary divisors is reducible to diagonal matrices under the same similarity transformation if and only if $A_iA_j=A_jA_i$ for all i and j.) The following will be shown to hold:

THEOREM. *If $\{A_i\}$ is an arbitrary set of nonzero $r\times s$ matrices, there are unitary matrices U and V of orders $r\times r$ and $s\times s$, respectively, such that $UA_iV=D_i$, D_i diagonal and real, if and only if $A_iA_j^{ct}=A_jA_i^{ct}$ and $A_j^{ct}A_i=A_i^{ct}A_j$ for all i and j.*

If two unitary matrices U and V exist such that $UA_iV=D_i$, D_i real for all i, then $D_iD_j^{ct}=D_iD_j^t=D_jD_i^t=D_jD_i^{ct}$ where the D_i are $r\times s$ diagonal matrices (that is, the only nonzero elements appear in the d_{ii} position). Therefore, $A_iA_j^{ct}=A_jA_i^{ct}$.

Conversely, let the relations $A_j^{ct}A_i=A_i^{ct}A_j$ and $A_iA_j^{ct}=A_jA_i^{ct}$ hold for all i, j. The proof is by induction.

(1) The theorem is true for a set of matrices of dimension $1\times s$, $A_i=[a_i', a_i'', \cdots, a_i^{(s)}]$. For there exist unitary matrices U and V such that[1] $UA_1V=[d_1', 0, \cdots, 0]$ for d_1' real and greater than 0 since $A_1\neq 0$. For if $UA_iV=[d_i', d_i'', \cdots, d_i^{(s)}]$, it follows from $A_i^{ct}A_1=A_1^{ct}A_i$ that $d_i''=d_i'''=\cdots=d_i^{(s)}=0$ and since $d_1'\cdot\bar{d}_i'=\bar{d}_1'\cdot d_i'$ and d_1' is real, $\bar{d}_i'=d_i'$. In the same way by means of the second of

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[1] Bull. Amer. Math. Soc. vol. 45 (1939) pp. 118–121. See also J. Williamson, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 920–922.