ON THE CONVEXITY OF MEAN VALUE FUNCTIONS

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1. **Introduction.** Let (a) denote a set $a_1, a_2, \cdots, a_n$ of n distinct positive numbers, $n \geqq 2$, with the subscripts ν labeled so that $a_\nu < a_{\nu+1}$ for $\nu = 1, \cdots, n-1$. Let (ξ) denote a set of positive numbers $\xi_1, \xi_2, \cdots, \xi_n$ with $\sum_{\nu=1}^n \xi_\nu = 1$. The mean value function $M_t(a, \xi) = (\sum_{\nu=1}^n \xi_\nu a_\nu^t)^{1/t}$, $t \neq 0$, $\pm\infty$; $M_0(a, \xi) = \prod_{\nu=1}^n a_\nu^{\xi_\nu}$; $M_{-\infty}(a, \xi) = \min_{\nu=1,2,\cdots,n} a_\nu$ and $M_{+\infty}(a, \xi) = \max_{\nu=1,2,\cdots,n} a_\nu$; is a continuous and strictly increasing function of t for $-\infty \leqq t \leqq +\infty$.[1] For given fixed sets (a) and (ξ), let $M(t)$ denote $M_t(a, \xi)$ and $\Lambda(t)$ denote $\log M_t(a, \xi)$. Each of the functions $M(t)$ and $\Lambda(t)$ has horizontal asymptotes and consequently at least one point of inflection. We shall show that these functions may have more than one inflection point, but shall show that $\Lambda(t)$ is a convex function of t in a neighborhood of $-\infty$, and a concave function of t in a neighborhood of $+\infty$. A sufficient condition will be obtained for $\Lambda(t)$ to be convex for all negative t, and one for $\Lambda(t)$ to be concave for all positive t. Finally, the applicability of the methods used to more general weighted sums will be considered briefly.

2. **Notations and fundamental formulae.** Let

$$f(t) = \log\left(\sum_{\nu=1}^{n} \xi_\nu a_\nu^t\right), \tag{1}$$

$$\eta_\nu(t) = \frac{\xi_\nu a_\nu^t}{\sum_{\nu=1}^{n} \xi_\nu a_\nu^t}, \quad \lambda_\nu = \log a_\nu \qquad (\nu = 1, 2, \cdots, n), \tag{2}$$

$$S_k = \sum_{\nu=1}^{n} \eta_\nu (\lambda_\nu)^k. \tag{3}$$

Then

$$\frac{d\eta_\nu}{dt} = \eta_\nu\lambda_\nu - \eta_\nu S_1, \quad \frac{dS_k}{dt} = S_{k+1} - S_1 S_k,$$

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[1] See Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge University Press, 1934, chap. 2, for the basic properties of the mean value function.