ZEROS OF THE HERMITE POLYNOMIALS AND WEIGHTS FOR GAUSS' MECHANICAL QUADRATURE FORMULA

ROBERT E. GREENWOOD AND J. J. MILLER

In the numerical integration of a function $f(x)$ it is very desirable to choose the set of values $\{x_i\}$ at which the function $f(x)$ is to be observed, for it is generally possible to obtain the same accuracy with fewer points when these points are especially selected. Gauss[1] gave such a proof for the case of the finite range $(-1, +1)$ and established that the "best" accuracy with n ordinates is obtained when the corresponding abscissae are the n roots of the Legendre polynomials, $P_n(x)=0$. For this case there obtains

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=1}^{n} \lambda_{i,n} f(x_{i,n}) \tag{1}$$

where the numbers $\{x_{i,n}\}$ are the zeros of $P_n(x)$ and where the numbers $\{\lambda_{i,n}\}$ are the Christoffel or Cotes numbers. Formula (1) is exact whenever $f(x)$ is a polynomial of degree $(2n-1)$ or less. Values of the zeros $\{x_{i,n}\}$ and the corresponding Christoffel numbers $\{\lambda_{i,n}\}$ for the Legendre polynomials for $n=1$ to $n=16$ have been tabulated by the Mathematical Tables Project.[2] The range of integration can be chosen to be any finite range (p, q) with suitable modification[2] of the zeros $\{x_{i,n}\}$ and the constants $\{\lambda_{i,n}\}$.

It is understood that while selection of the abscissae $\{x_{i,n}\}$ is very desirable for theoretical reasons, it may not always be practicable to measure the ordinates of $f(x)$ at these values.

For the infinite range $(-\infty, +\infty)$ a similar situation holds for the Hermite polynomials. These may be defined by the relation

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} \frac{d^n(e^{-x^2})}{dx^n} \\ &= (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} \pm \cdots . \end{aligned} \tag{2}$$

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[1] C. F. Gauss, *Methodus nova integralium valores per approximationem inveniendi*, Werke, vol. 3, pp. 163–196.

[2] A. N. Lowan, Norman Davids and Arthur Levenson, *Table of the zeros of the Legendre polynomials of order* 1–16 *and the weight coefficients for Gauss' mechanical quadrature formula*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 739–743.