GENERALIZATION OF AN INEQUALITY OF HEILBRONN AND ROHRBACH

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Let $a_1, \cdots, a_m$ be positive integers and

$$(1)\quad T(a_1, \cdots, a_m) = \begin{cases} 1 - \sum_{\mu_1=1}^{m} \frac{1}{a_{\mu_1}} + \sum_{\mu_1=2}^{m} \sum_{\mu_2=1}^{\mu_1-1} \frac{1}{\{a_{\mu_1}, a_{\mu_2}\}} - \cdots \\ \qquad + \frac{(-1)^m}{\{a_1, \cdots, a_m\}} & \text{for } m > 0, \\ 1 & \text{for } m = 0, \end{cases}$$

where $\{u_1, \cdots, u_r\}$ denotes the least common multiple of $u_1, \cdots, u_r$. H. A. Heilbronn[1] and H. Rohrbach[2] proved that

$$(2)\quad \begin{aligned} T(a_1, \cdots, a_m) &\geqq \left(1 - \frac{1}{a_1}\right) \cdot \cdots \cdot \left(1 - \frac{1}{a_m}\right) \\ &= T(a_1) \cdot \cdots \cdot T(a_m). \end{aligned}$$

The object of this paper is to prove the following generalization of (2):

$$(3)\quad T(a_1, \cdots, a_m, b_1, \cdots, b_n) \geqq T(a_1, \cdots, a_m)T(b_1, \cdots, b_n) \qquad \text{for } m \geqq 0,\ n \geqq 0.$$

$T(a_1, \cdots, a_m)$ may be interpreted as the density of the set S of all positive integers not divisible by any a_μ, that is,

$$T(a_1, \cdots, a_m) = \lim_{z \to \infty} z^{-1} M(z),$$

where $M(z)$ is the number of elements of S not exceeding z.

For the proof of (3) we require the following lemma.

LEMMA. *If* $k \geqq 0$, $l \geqq 0$, *and* $(d, v_\lambda) = 1$ *for* $\lambda = 1, \cdots, l$, *then*

$$T(du_1, \cdots, du_k, v_1, \cdots, v_l) = \frac{1}{d} T(u_1, \cdots, u_k, v_1, \cdots, v_l) + \left(1 - \frac{1}{d}\right) T(v_1, \cdots, v_l).$$

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[1] *On an inequality in the elementary theory of numbers*, Proc. Cambridge Philos. Soc. vol. 33 (1937) pp. 207–209.

[2] *Beweis einer zahlentheoretischen Ungleichung*, J. Reine Angew. Math. vol. 177 (1937) pp. 193–196.