RINGS WITH A POLYNOMIAL IDENTITY

IRVING KAPLANSKY

1. Introduction. In connection with his investigation of projective planes, M. Hall [2, Theorem 6.2][1] proved the following theorem: a division ring D in which the identity

$$(1) \qquad (xy - yx)^2 z = z(xy - yx)^2$$

holds is either a field or a (generalized) quaternion algebra over its center F. In particular, D is finite-dimensional over F, something not assumed a priori. The main result (§2) in the present paper is the following: if D satisfies *any* polynomial identity it is finite-dimensional over F. There are connections with other problems which we note in §§3, 4.

2. Proof of finite-dimensionality. Let A be an algebra (no assumption of finite order) over a field F. We denote by $F[x_1, \cdots, x_r]$ the free algebra generated by r indeterminates over F. We say that A satisfies a polynomial identity if there exists a nonzero element f in $F[x_1, \cdots, x_r]$ such that $f(a_1, \cdots, a_r) = 0$ for all a_i in A.

LEMMA 1.[2] *If A satisfies any polynomial identity, then it satisfies a polynomial identity in two variables.*

PROOF. Suppose A satisfies the equation $f(x_1, \cdots, x_r) = 0$. Replacing x_i by $u^i v$ we obtain the equation $g(u, v) = 0$, with g a polynomial which is not identically zero.

LEMMA 2. *If A satisfies any polynomial identity, it satisfies a polynomial identity which is linear in each variable.*

PROOF. Suppose A satisfies $f(x_1, \cdots, x_r) = 0$ and that f is not linear in x_1. Then

$$f(y + z, x_2, \cdots, x_r) - f(y, x_2, \cdots, x_r) - f(z, x_2, \cdots, x_r) = 0$$

is satisfied by A. This is a polynomial (in $r+1$ variables), not identically zero, and with degree in y and z lower than the degree of f in x_1. By successive steps of this kind we reach a polynomial linear in all variables.

Presented to the Society, September 3, 1947; received by the editors August 20, 1947.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] Cf. [7, Satz 2].