A NOTE ON LACUNARY POLYNOMIALS

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1. **Introduction.** In the present note we shall give an elementary derivation of some new bounds for the p smallest (in modulus) zeros of the polynomials of the lacunary type

$$(1.1)\quad f(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots + a_{n_k} z^{n_k},$$
$$a_0 a_p \neq 0, \quad 0 < p = n_0 < n_1 < \cdots < n_k.$$

This will be done by the iterated application, first, of Kakeya's Theorem[1] that, if a polynomial of degree n has p zeros in a circle C of radius R, its derivative has at least $p-1$ zeros in the concentric circle C' of radius $R' = R\phi(n, p)$; and, secondly, of the specific limits

$$(1.2)\qquad \phi(n, p) \leqq \csc\left[\pi/2(n - p + 1)\right],$$

$$(1.3)\qquad \phi(n, p) \leqq \prod_{j=1}^{n-p} (n + j)/(n - j)$$

furnished by Marden[2] and Biernacki[3] respectively.

2. **Derivation of the bounds.** An immediate corollary to Kakeya's Theorem is:

THEOREM I. *If the derivative of an nth degree polynomial $P(z)$ has at most $p-1$ zeros in a circle Γ of radius ρ, then $P(z)$ has at most p zeros in the concentric circle Γ' of radius $\rho' = \rho/\phi(n, p+1)$.*

We shall use Theorem I to prove the following theorem.

THEOREM II. *If all the zeros of the polynomial*

$$(2.1)\quad \begin{aligned} f_0(z) = {} & n_1 n_2 \cdots n_k a_0 + (n_1 - 1)(n_2 - 1) \cdots (n_k - 1) a_1 z \\ & + \cdots + (n_1 - p)(n_2 - p) \cdots (n_k - p) a_p z^p \end{aligned}$$

lie in the circle $|z| \leqq R_0$, at least p zeros of polynomial (1.1) *lie in the circle*

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[1] S. Kakeya, Tôhoku Math. J. vol. 11 (1917) pp. 5–16.

[2] M. Marden, Trans. Amer. Math. Soc. vol. 45 (1939) pp. 335–368. See also M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, to be published as a volume of Mathematical Surveys.

[3] M. Biernacki, Bull. Soc. Math. France (2) vol. 69 (1945) pp. 197–203.