ON SOME CRITERIA OF CARLEMAN FOR THE COMPLETE CONVERGENCE OF A J-FRACTION

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1. Introduction. Carleman [1, pp. 214–215][1] derived, from his theory of integral equations, a number of criteria for the complete convergence of a real J-fraction

$$(1.1)\qquad -\mathop{K}_{p=1}^{\infty} \frac{-a_{p-1}^2}{b_p + z} \qquad (a_0 = 1,\ a_p \neq 0).$$

The most important one of these criteria states that the J-fraction is completely convergent if the series $\sum (1/c_{2p})^{1/2p}$ diverges, where $c_0, c_1, c_2, \cdots$ are the coefficients in the power series $\sum (c_p/z^{p+1})$ associated with the J-fraction. In [2] Carleman gave an algebraic proof of this theorem for the case where $b_p = 0$, $p = 1, 2, 3, \cdots$. The present note contains an algebraic proof for the general case, and some remarks concerning the other criteria of Carleman, especially with reference to their application to J-fractions with arbitrary complex coefficients.

2. The determinate and indeterminate cases. Let $a_0 = 1$, $a_p \neq 0$, b_p, $p = 1, 2, 3, \cdots$, be complex constants, and consider the system of linear equations

$$(2.1)\qquad -a_{p-1}x_{p-1} + (b_p + z)x_p - a_p x_{p+1} = 0, \qquad p = 1, 2, 3, \cdots.$$

Since the a_p are not zero, these equations determine $x_2, x_3, x_4, \cdots$ uniquely in terms of arbitrarily chosen initial values x_0, x_1. If $x_0 = -1$, $x_1 = 0$, let $x_p = X_p(z)$, and if $x_0 = 0$, $x_1 = 1$, let $x_p = Y_p(z)$. Then, $X_{p+1}(z)/Y_{p+1}(z)$ is the pth approximant of the J-fraction (1.1). If the infinite series $\sum |X_p(z)|^2$ and $\sum |Y_p(z)|^2$ both converge for one value of z, then they converge for every value of z [7, p. 120]. We may accordingly distinguish two cases for a J-fraction (1.1) with complex coefficients. In the *indeterminate case*, both the infinite series

$$(2.2)\qquad \sum |X_p(0)|^2, \qquad \sum |Y_p(0)|^2$$

are convergent. In the *determinate case*, at least one of these infinite series is divergent. A real J-fraction is completely convergent if and only if the determinate case holds. This is also the case of a determinate moment problem [4].

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[1] Numbers in brackets refer to the bibliography at the end of the paper.