THE MULTIPLICATIVE COMPLETION OF SETS OF FUNCTIONS

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1. **Introduction.** A set $\{f_n(x)\}_1^\infty$ of functions of $L^2(a, b)$, where (a, b) is finite or infinite, is called complete if $g(x) \in L^2$ and $\int_a^b f_n(x)g(x)dx = 0$, $n = 1, 2, \cdots$, imply that $g(x) = 0$ almost everywhere on (a, b); a well known equivalent property ("closure") is that every element of L^2 can be approximated in the L^2 metric by finite linear combinations of the $f_n(x)$.

Suppose that $\{f_n(x)\}$ is not complete. It will sometimes be possible to find a function $m(x)$ such that the set $\{m(x)f_n(x)\}$ is complete. This can also be considered as completeness after a change of weight function or a change of measure; but we shall not attempt to consider the most general change of measure here. We give some results on when a set can or cannot be completed by multiplication; the problem of finding necessary and sufficient conditions is left open.

We first state our results.

THEOREM 1. *If $\{f_n(x)\}_1^\infty$ is an orthonormal set which is not complete, but can be completed by the addition of a finite number of functions to the set, then there is a bounded measurable function $m(x)$ such that $\{m(x)f_n(x)\}_1^\infty$ is complete.*

The condition of Theorem 1, while necessary, is not sufficient, as Theorem 2 shows.

THEOREM 2. *The orthogonal set $\{e^{-x/2}L_{2n}(x)\}_0^\infty$, where $L_{2n}(x)$ is the 2nth Laguerre polynomial, cannot be completed on $(0, \infty)$ by the addition of a finite number of functions, but is completed on multiplication by $m(x) = e^{-x/2}$.*

Our next three theorems give examples of sets which cannot be completed by multiplication.

THEOREM 3. *A set of even functions cannot be completed by multiplication by an integrable function in any interval containing 0.*

THEOREM 4. *The set $\{e^{2inx}\}_{-\infty}^{\infty}$ cannot be completed in $(-\pi, \pi)$ by multiplication by an integrable function.*

THEOREM 5. *The set $\{x^{\lambda_n}\}$, where $\lambda_n > 0$, $\sum 1/\lambda_n < \infty$, cannot be completed in any interval by multiplication by a continuous function.*

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