THE RANGE OF A VECTOR MEASURE

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The purpose of this note is to prove that the range of a countably additive finite measure with values in a finite-dimensional real vector space is closed and, in the non-atomic case, convex. These results were first proved (in 1940) by A. Liapounoff.[1] In 1945 K. R. Buch (independently) proved the part of the statement concerning closure for non-negative measures of dimension one or two.[2] In 1947 I offered a proof of Buch's results which, however, was correct in the one-dimensional case only.[3] In this paper I present a simplified proof of Liapounoff's results.

Let X be any set and let S be a σ-field of subsets of X (called the measurable set of X). A *measure* μ (or, more precisely, an N-dimensional measure $(\mu_1, \cdots, \mu_N)$) is a (bounded) countably additive function of the sets of S with values in N-dimensional, real vector space (in which the "length" $|\xi_1| + \cdots + |\xi_N|$ of a vector $\xi = (\xi_1, \cdots, \xi_N)$ is denoted by $|\xi|$). The measure $(\mu_1, \cdots \mu_N)$ is *non-negative* if $\mu_i(E) \geqq 0$ for every $E \in S$ and $i = 1, \cdots, N$.

For a numerical (one-dimensional) measure μ_0, $\mu_0^*(E)$ will denote the total variation of μ_0 on E; in general, if $\mu = (\mu_1, \cdots, \mu_N)$, μ^* will denote the non-negative measure $(\mu_1^*, \cdots, \mu_N^*)$. The length $|\mu^*| = \mu_1^* + \cdots + \mu_N^*$ is always a non-negative numerical measure.[4]

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[1] *Sur les fonctions-vecteurs complètement additives*, Bull. Acad. Sci. URSS. Sér. Math. vol. 4 (1940) pp. 465–478. In a subsequent paper with the same title, in the same journal (vol. 10 (1946) pp. 277–279), Liapounoff gave a very elegant example to show that neither convexity nor closure can always be asserted in the infinite-dimensional case. Related questions for finitely additive measures have been discussed by A. Sobczyk and P. C. Hammer, *The ranges of additive set functions*, Duke Math. J. vol. 11 (1944) pp. 847–851.

[2] *Some investigations of the set of values of measures in abstract space*. K. Danske Videnskabernes Selskab, Mathematish-Fysishe Meddelelser vol. 21.

[3] *On the set of values of a finite measure*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 138–141. I am indebted to Professors Børge Jessen and Jean Dieudonné for calling my attention to the fact that the statement and proof of Lemma 5 of that paper are wrong. The error in the proof is the implicit assumption that if S is covered by sets of the form $U \cap V$ then the intersection of any U that occurs with any V that occurs is also a set of the covering. It is not difficult to show that the conclusion of Lemma 5 holds if and only if every closed set of either of the two given topologies is compact with respect to the other. It follows that if, for instance, the given topologies are Hausdorff then the conclusion holds if and only if they are identical.

[4] For the notion of total variation, as well as all other concepts and results of elementary measure theory, see S. Saks, *Theory of the integral*, Warsaw, 1937, chaps. 1 and 2.